Notes on explaining mathematics

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These notes aim to help students express themselves better when discussing mathematics in English.

Common mathematical phrases

This section looks at some phrases which are common in mathematical communication, and discuss how to use them correctly and how to interpret them.

The goals, the setting and the audience

Depending on the audience and the setting different phrases may be appropriate, or may be interpreted in different ways. It is therefore important to clearly define each.

There are two common settings.

- Written mathematics. In this case the audience can go at their own pace, and can easily look up anything they have forgotten, or supply small arguments.
- A presentation. In this case it is much harder for the audience to take in new information, and the logical structure needs to be made clear. The speaker has more tools to communicate, including body language and actions.

Here are examples of goals, and situations where they might apply. Note that the goals may change even in the same presentation.

- the writer/speaker wants the reader/listener to understand everything in detail (*e.g.* a lecturer teaching students, a textbook author);
- the writer/speaker wants the reader/listener to get an idea of why the results are true, but do not need to explain the details (*e.g.* a short research presentation);
- the writer/speaker wants to convince the reader/listener that they understand in detail (*e.g.* a student in an examination).

The words and phrases below give a different impression depending on the context of the writing/presentation.

Exercise. *Discuss how the word* obvious *could be used by each of the following, and how it is likely to be interpreted by their audience: (a) a research mathematician explaining a generalisation of a well known result; (b) a lecturer explaining a proof to a group of students; (c) a research mathematician explaining a proof of a famous unsolved problem; (d) a student presenting the results in their thesis.*

Describing a logical step

Very often we want to communicate a logical implication, such as "if P then Q"; to help the reader/listener one can add words or phrases: "if P then Q because..." or "if P then clearly Q". The word *because* (which can be replaced by *since* or *as*, for example) is followed by an explanation, and most of the time there is no problem. On the other hand, the word *clearly* (see below for more examples) gives a signal to the reader/listener, and is not always followed by an explanation. The following words occur very frequently in mathematical communication, normally in a proof and in a similar way to the "if P then clearly Q" example, but they are often used incorrectly or misunderstood. The notes explain correct usage.

trivial Can be used in several different ways.

- A statement or step in a proof which does not require any further explanation at all (when proving $n^n \geq n!$ for all $n \in \mathbb{N}$ by induction the base case $n = 1$ is trivial);
- some logical implications are called *trivial*, for example the implication $P \implies Q$ in the case that Q is a true statement;
- a very simple (or perhaps uninteresting) example of an object with a certain property: a trivial solution of an equation, the trivial group, a trivial subgroup. (In this case *trivial* does not describe a logical step.)

clearly/it is clear that In mathematical writing this means that the reader should be able to understand the logical step being taken without any help. Different authors may have very different standards for exactly how easy the process of understanding this step should be. In a research presentation this may be used to mean "if one has time to understand the material properly then it is clear that".

obviously/it is obvious that The same comments as for *clearly* above apply.

it is easy to see that The same comments as for *clearly* above apply; may indicate that a little more work is expected from the reader.

it is routine that Means there is some work to be done, but the work does not need anything except standard tools. If only one extra tool is required then *routine* is also used, for example "a routine application of the Hahn–Banach Theorem".

apparently Formally this word has a similar meaning to *clearly*; it should not be used in mathematics because in common usage it may indicate that the speaker does not know the reason.

by inspection This is used in two ways: inspection of a formula shows some property; inspection of the proof shows it can be generalised.

handwaving When presenting mathematics it is common to give informal arguments, for example using diagrams, physical intuition, or by discussing only a special case, to give listeners an idea of why a mathematical statement is true (this can either be supplementary to a proper proof or because it is not possible to present the proper proof due to time, difficulty, *etc.*) To me *handwaving* is usually negative, as it means the speaker is avoiding difficult details.

we can see that Signals that there is a logical step, but does not indicate the difficulty of the step, or anything else.

it is possible to show that This indicates that the writer/speaker does not want to explain the reason.

In good mathematical writing these words give some feedback to the reader. For example, if a textbook for undergraduates uses *clearly* this means that a student reading the textbook should find the statement clear, based on what they are assumed to know. A well-written piece of mathematics also uses these words to avoid getting bogged down in details which distract from the main point.

Students should be especially careful with these phrases because a student is usually writing/speaking to demonstrate that they themselves understand. An unscrupulous student may try to hide that they do not understand well behind words like *clearly*, hoping that when their work is graded it will be believed. It is also quite common that a student mistakenly uses one of these words because they have made an honest mistake — they honestly believe something is easy but have missed some details.

Unfortunately these words are also misused by other authors/speakers. It is easy for the author of a textbook to think a step is *clear*, but to a reader who is relatively new to the material the step is not at all clear, for example because they have not yet internalised the basic techniques of the field.

It is often helpful to include an extra phrase with words like *clearly* or *obvious* to help the reader, for example: "it is obvious *from equation (2.4)* that f is well-defined". The phrase in italics provides the reader/listener enough information to understand the argument; without it the reader/listener may have no idea what is happening.

Unfortunately these words are not always used this way — a lazy or careless writer might write “it is obvious that...” because it is obvious to them, at the moment they wrote that sentence. This is not very considerate to the reader.

Finally, if you are tempted to use one of these make sure that there really is no subtlety. A useful test is to imagine being asked to explain the step which is “obvious” or “clear”.

Describing a choice or definition

These words can sometimes be confusing for students, because they are often used without further explanation.

arbitrary An *arbitrary* choice means that the object being chosen has no special properties.

natural Formally (in Category Theory) a *natural* choice this means that there are no arbitrary choices involved (this can be made rigorous using equations). It is also used informally to mean that a choice is obvious or sensible, often due to an analogy or motivating idea.

without loss of generality This is used when making an assumption which appears to be limiting, but the assumption is actually not limiting, either because trivial modifications to the argument solve the other cases, or because the argument can be reduced to the case which is assumed.

well defined Usually indicates that there is something to be checked. Most often this involves a definition which uses an arbitrary choice, and *well-definedness* means that any allowable choice is acceptable. The classic example involves a set X and an equivalence relation $\sim$ on X; one defines a “function” $f : X/\sim \to Y$ by $f([x]) :=$ (some formula involving x). To show f is well defined one must check that if $[x] = [y]$ then using y in the formula for f gives the same result as using x.

continue in this fashion A pattern has been established and should be followed for as long as possible, or as long as indicated.

Other

we can say that Often used by students, but it is weak and makes it seem that the writer/speaker does not understand. This phrase should be replaced by a proper explanation (“we can say that the function is continuous” should be replaced by “the function is continuous because...”) or sometimes by a definite statement and an indication the reasoning is to be skipped (“it is possible to show that the function is continuous”).

up to Saying that a statement P is true *up to* some property means that the statement is true except for the freedom given by the property (for example, every finite-dimensional vector space over $\mathbb{R}$ is, up to isomorphism, of the form $\mathbb{R}^n$ for some $n \in \mathbb{N}$).

well known The author/speaker believes that the relevant fact is likely to be known to the reader/listener, or perhaps should be known to the reader/listener.

Reading mathematics aloud

When presenting mathematics one will often have to read an equation, calculation, or some other collection of mathematical symbols. It is tempting to just read the symbols exactly as they appear, but this is not very helpful for the audience (they can already read the symbols as they appear on the screen). The goal should be to help the audience grasp what the symbols mean and to explain why they are important.

How many correct ways can you come up with to read the following mathematical notations? Explain how you are interpreting the symbols.

(1) $f(x)$

(2) $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$

(3) y^2

(4) g^{-1}

(5) $T^{-1}(\{0\})$

(6) ab or $a \times b$

(7) $2\pi/n$ or $\frac{2\pi}{n}$

(8) $f \circ g$

(9) $ST = TS$

(10) $f(x) = c$ for all $x \in \mathbb{R}$

Solutions

(1) "f of x"

(2) "the operation $*$ is associative"

(3) "y squared" or "y to the power two"

(4) "g inverse" or "the inverse of g" or "g to the power minus one"

(5) "the inverse image of 0 (under T)" or "the kernel of T" (if T is linear)

(6) "a times b" or "a multiplied by b" or "the product of a and b"

(7) "two pi over n" or "two pi divided by n"

(8) “g composed with f” or “f composed with g” or “g then f”

(9) “S and T commute”

(10) “f is constant” or “f is a constant function”

Some ideas to help the audience understand or show own understanding

When explaining some mathematics the goal is often to help the audience understand the topic, so one needs to consider carefully how the presentation will look to an audience member. Occasionally (often for students) the situation is different, for example in an oral examination the student explains to the examiner with the goal of convincing the examiner that the student understands.

The following points contain some ideas to consider when preparing for the above situations.

background Identify the background material required to understand the presentation. Decide if the audience needs this background explained in detail, if they need only brief reminders, or if it can be skipped.

key points Each time you introduce something new — a definition, theorem statement, proof, etc. — use speaking to emphasise the key point, either using tone or by saying explicitly. The key point could be the important part of the definition, the main conclusion in the theorem, or the most difficult step in the proof.

explain in different ways If available then two different perspectives gives the audience double the chance to grasp the topic. For a simple example, the Binomial Theorem can be visualised geometrically, counting the extra faces, edges and corners of n-dimensional cubes, or combinatorially, using the distributive property to expand the product $(a + b)^n$.

simple examples Often a (well-chosen) simple example helps the audience to grasp the main idea of a definition, theorem, etc. Aim for an example which shows the key points and which the audience already knows, so that there is no background, new notation, etc. to obscure the key points. The ability to identify a suitable example is a good demonstration of understanding.

formal and informal By themselves the formal statements of results can be difficult for the audience to process. It is very helpful for the audience to say informally what the statement means. For example “this says that every squiggle-function is close to an elementary squiggle-function”, “the Fundamental Theorem of Arithmetic says primes are the building blocks of integers under multiplication, and each number has a unique way to build it”.

analogies An analogy is an extremely informal, perhaps even to the point of silliness, statement which may help the audience understand a key point. A good analogy can be striking and memorable, but not at the price of being mathematically misleading.

diagrams A diagram gives the audience something visual to remember from the talk; they can be used in several different ways. If one defines a complicated condition on a function then it can be illustrated by drawing how the graph should look in the case of a function $\mathbb{R} \to \mathbb{R}$. Venn diagrams can be useful to illustrate relationships such as containment, intersection and union. Commutative diagrams or other graph-like diagrams can be very useful to clarify arguments involving lots of different compositions of functions, or the relationship between logical implications, for example.

anticipate questions After preparing the presentation think carefully about what are natural questions from the audience and prepare an answer. This exercise is useful for closing small gaps in understanding, catching small mistakes in the presentation, and often makes for a more detailed presentation.

Writing versus presenting

At the beginning we mentioned briefly that writing and presenting have different requirements, here are some areas to consider.

formality Writing is almost always more formal than presentations. In written mathematics the standard is to write formally, stripping away almost all the informal statements. As discussed elsewhere, presentations can include more informal speaking, though a presentation with nothing formal (like a proper definition and theorem) vomes across as very thin, lacking mathematical depth.

memory In a presentation the information disappears quickly, and the audience cannot be expected to remember anything. One needs to separate ideas clearly, so each idea is used on the same slide it is introduced, or have full reminders, or 'slogans' one can repeat. In written mathematics the reader can go back and forward through the text as much as they want, so it is not so important to separate ideas very clearly (though it is still good practice to do so). Full reminders are not necessary in writing, just helpful cues to make life easier for the reader: good cross-references to numbered definitions, theorems, etc, are essential; some other signals can also be useful, for example "similarly *to the calculation above*", or "following the analogy *from the introduction*".

speed In a talk there is a time limit, therefore a limit on the amount of new information that can be included. For the reader of a paper they have as much time as they want (within reason) to absorb new information, assuming the writing is interesting enough.

calculations Long calculations in a talk are very boring and it is unlikely the audience will remember much. Instead of showing $a = b = c = \cdots = x$ it is better to state $a = x$ and explain the key step is $m = n$, the others are routine. In writing the reader can skim routine steps by themselves if they see what to do, but it is better to include most of the steps.

emphasis In a presentation one can — in fact should — use their voice to emphasise important points: the key step in a calculation, the main idea in a theorem, the important conclusion of a theorem, etc. If these words are simply written down then the tone is lost, so the reader cannot tell where the emphasis is. This emphasis must be added back in writing, with short, clear comments before or after definitions, theorems, proofs, etc. or in a numbered remark for important comments.

details Often details are hidden in presentations, even in an in-depth research talk; there are simply too many details, so the audience needs to trust that the speaker is not tricking them, though they can ask if they need more information. In writing this kind of interaction between author and reader is not easily available, so there is more potential for the reader to get confused. It is therefore important, and also expected by others, to include all details somewhere in the article unless you explicitly say you are not including details and give a good reason why.

Mathematical writing: some basic rules

grammar Use proper punctuation, including full stops, commas, colons, semicolons, parentheses and paragraphs.

clarity Aim for crisp, clear sentences. Each sentence should make exactly one relevant point. It will take some editing to get to this stage.

precision Check your work to make sure there is no ambiguity and it is easy to understand.

symbols and equations Sentences never begin with mathematical symbols. Equations are punctuated as part of a sentence. Do your best to make calculations easy to read.