The Cantor set

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The Cantor set is a topological space, constructed as a subspace of $\mathbb{R}$, with many interesting and surprising properties. It began appearing in the mathematical literature before 1880, in connection with developments in Topology and Integration, apparently first mentioned by Smith [Smi], later by du Bois-Reymond [dBR], Volterra [Vol], and Cantor [Can]. It is now named after Cantor.

1 Construction and basic properties

To begin we introduce some notation. Suppose that $A \subset \mathbb{R}$ can be written as the union of finitely many closed, pairwise-disjoint intervals

$$A = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n].$$

For a natural number $b \geq 3$ define ${*_b}A$ to be the union of the upper and lower $1/b$ of each interval of A:

$$\begin{aligned} {*_b}A := \left[a_1, a_1 + \frac{1}{b}(b_1 - a_1)\right] \bigcup \left[a_1 + \frac{b-1}{b}(b_1 - a_1), b_1\right] \bigcup \cdots \\ \cdots \bigcup \left[a_n, a_n + \frac{1}{b}(b_n - a_n)\right] \bigcup \left[a_n + \frac{b-1}{b}(b_n - a_n), b_n\right]. \end{aligned}$$

Clearly ${*_b}A$ is again the union of finitely many closed, pairwise-disjoint intervals, so we may apply the same construction again, forming a sequence

$$A, \; {*_b}A, \quad {*_b}({*_b}A) = {*_b^2} A, \quad {*_b^3}A, \ldots$$

of unions of increasingly many disjoint intervals, of shrinking length.

We will only use this construction in the case our initial set A is the unit interval $[0, 1]$. In the next result we collect some easy observations about this operation.

Lemma 1.1. *Let $n \in \mathbb{N}$. The set* $*_b^n[0,1]$*:*

(i) is the union of 2^n disjoint closed intervals;

(ii) consists of intervals of length b^{-n};

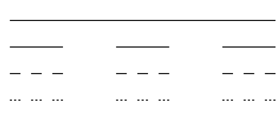
(iii) is constructed from $*_b^{n-1}[0,1]$ by removing 2^{n-1} disjoint open intervals, each of length $(b-2)/b^n$;

(iv) consists of intervals with left endpoints in the set

$$L_n^b = \left\{ \frac{c_1}{b} + \frac{c_2}{b^2} + \cdots + \frac{c_n}{b^n} : c_i \in \{0, b-1\} \right\}.$$

Proof. Points (i)–(iii) are immediate from the definition. For (iv) we proceed by induction. It is clear that the set of left endpoints of intervals in $*_b^1[0,1]$ is $\{0, \frac{b-1}{b}\}$ so the claim is true for $k = 1$. Now suppose the claim holds for $n = k$. By definition of $*_b^{k+1}[0,1]$ from $*_b^k[0,1]$ each interval of $*_b^k[0,1]$, hence each element $c \in L_k^b$, determines two intervals of $*_b^{k+1}[0,1]$, hence two elements of L_{k+1}^b. The first of these elements is c again, while the second is $c + \frac{b-1}{b^{k+1}}$. This completes the induction and establishes the claim. □

It is most common to work with $b = 3$ or $b = 10$. Here is an illustration of the first few steps of the process in the case $b = 3$.



Definition 1.2. *The* Cantor set (in base b) *is defined as*

$$\text{Cantor}_b := \cap_{n \in \mathbb{N}} *_b^n[0,1].$$

1.1 Properties of base b representations

Our next aim is to answer the question *what is in* Cantor_b*?* To discuss this first we recall some facts about base b representations of numbers. If $b \geq 2$ is a natural number then any $x \in \mathbb{R}$ can be represented as

$$x = \sigma \sum_{k \in \mathbb{Z}} a_k b^k$$

where $\sigma \in \{+,-\}$ is the sign, $a_k \in \{0,1,2,\ldots,b-1\}$ and there is $n_0 \in \mathbb{Z}$ such that $a_k = 0$ for all $k > n_0$, so the infinite sum clearly converges. We write this number using *positional notation* as

$$x = \sigma a_{n_0} a_{n_0-1} \ldots a_2 a_1 a_0 . a_{-1} a_{-2} a_{-3} \ldots_{[b]}.$$

Note that to avoid ambiguity in this notation when $b \geq 10$ one may need to introduce additional symbols for the digits bigger than 9, since for example $2113_{[b]} = 2b^3 + b^2 + b + 3$; the number $2b^2 + 11b + 3$ must be written $2\chi 3_{[b]}$ where χ represents the eleventh base b digit. It is important to remember that the representation of a number in this way is not unique (for example $1_{[10]} = 0.999\ldots_{[10]}$); this will be investigated in Lemma 1.4.

We will only be discussing numbers in the interval $[0,1]$, so it is convenient to make a slight change to the above convention and represent a number $x \in [0,1]$ in base b as

$$x = \sum_{k\in\mathbb{N}} c_k b^{-k} = 0.c_1 c_2 c_3 \ldots_{[b]}$$

(so $c_k = a_{-k}$).

Remark 1.3. *Suppose $x = \sum_{k\in\mathbb{N}} c_k b^{-k}$ and $c_1 = c_2 = \cdots = c_m = 0$, so in positional notation $x = 0.00\ldots 0c_{m+1}c_{m+2}\ldots_{[b]}$. In this case the largest x can be is when the non-zero c_k are all equal to $b-1$, so*

$$x = \sum_{k\in\mathbb{N}} c_k b^{-k} \leq \sum_{k=m+1}^{\infty} \frac{b-1}{b^k} = \frac{b-1}{b^{m+1}} \sum_{l=0}^{\infty} \frac{1}{b^l} = \frac{b-1}{b^{m+1}} \frac{1}{1-\frac{1}{b}} = \frac{b-1}{b^{m+1}} \frac{b}{b-1} = \frac{1}{b^m}.$$

On the other hand, if $x = \sum_{k\in\mathbb{N}} c_k b^{-k}$ and $c_k = 0$ for all $k > m$, but $c_m \neq 0$, then the smallest x can be is when $c_m = 1$ and $c_k = 0$ for all $k \in \mathbb{N} \setminus \{1\}$, so $x \geq 1/b^m$.

Lemma 1.4. *Let $b \geq 2$ be a natural number, $B = \{0,1,2,\ldots,b-1\}$ and $B^{\mathbb{N}}$ the collection of all sequences of elements of B. The function*

$$\beta : B^{\mathbb{N}} \to [0,1]; \ \beta\big((c_k)_{k\in\mathbb{N}}\big) := \sum_{k\in\mathbb{N}} \frac{c_k}{b^k}$$

has the following properties:

(a) β is surjective;

(b) for $c, d \in B^{\mathbb{N}}$ with $c \neq d$ the following are equivalent, characterising when β fails to be injective:

(i) $\beta(c) = \beta(d)$*;*

(ii) there exists $m \in \mathbb{N}$ such that $c_n = d_n$ for all $n < m$, $c_m - d_m = 1$, and $c_n = 0$ for all $n > m$ while $d_n = b - 1$ for all $n > m$ (or vice-versa).

In particular, an element of $[0, 1]$ has either one or two preimages.

Proof. (a) Let $x \in [0, 1)$. Choose $c_1 \in B$ to be the largest element for which $\frac{c_1}{b} \leq x$. Define inductively (with the help of Remark 1.3) c_{n+1} to be the largest element of B for which

$$0 \leq x - \sum_{k=1}^{n+1} \frac{c_k}{b^k} < \frac{1}{b^{n+1}}. \tag{1}$$

This gives a well-defined sequence $c = (c_k)_{k \in \mathbb{N}}$ and it is immediate from the definition of c and equation (1) that

$$\left| x - \beta(c) \right| = \lim_{n \to \infty} x - \sum_{k=1}^{n} \frac{c_k}{b^k} < \lim_{n \to \infty} \frac{1}{b^n} = 0$$

(we can remove the absolute value because the partial sum is by definition no greater than x). Hence $x = \beta(c)$.

For $x = 1$ consider the sequence $c \in B^{\mathbb{N}}$ with $c_k = b - 1$ for all $k \in \mathbb{N}$. Then

$$\beta(c) = \sum_{k \in \mathbb{N}} \frac{b-1}{b^k} = (b-1)\frac{1}{b-1} = 1.$$

(b) (ii) $\Longrightarrow$ (i) Suppose that c and d are as in (ii), so

$$\begin{aligned} \beta(d) = \sum_{k \in \mathbb{N}} \frac{d_k}{b^k} = \sum_{k=1}^{m-1} \frac{d_k}{b^k} + \frac{d_m}{b^m} + \sum_{l=m+1}^{\infty} \frac{d_l}{b^l} &= \sum_{k=1}^{m-1} \frac{c_k}{b^k} + \frac{d_m}{b^m} + \sum_{l=m+1}^{\infty} \frac{b-1}{b^l} \\ &= \sum_{k=1}^{m-1} \frac{c_k}{b^k} + \frac{d_m}{b^m} + \frac{1}{b^m} \\ &= \sum_{k=1}^{m-1} \frac{c_k}{b^k} + \frac{d_m + 1}{b^m} \\ &= \sum_{k=1}^{m} \frac{c_k}{b^k} = \beta(c), \end{aligned}$$

using Remark 1.3.

(i) $\implies$ (ii) Suppose $\beta(c) = \beta(d)$ but $c \neq d$ and let $m \in \mathbb{N}$ be the smallest index for which $c_m \neq d_m$. Without loss of generality we may assume $c_m > d_m$. Then $\beta(c) = \beta(d)$ implies

$$\frac{c_m - d_m}{b^m} + \sum_{k=m+1}^{\infty} \frac{c_k}{b^k} = \sum_{l=m+1}^{\infty} \frac{d_l}{b^l}. \tag{2}$$

We have $\frac{c_m - d_m}{b^m} \geq \frac{1}{b^m}$ and by Remark 1.3 also $\sum_{l=m+1}^{\infty} \frac{d_l}{b^l} \leq \frac{1}{b^m}$. Hence the only way for (2) to hold is if $c_m - d_m = 1$, $c_n = 0$ for all $n > m$, and $d_n = b - 1$ for all $n > m$. We have shown that the only two possible preimages of $x \in [0, 1]$ are the ones given in (ii). □

The above lemma can be extended to give a surjection onto $\mathbb{R}$, but the definition of the domain needs to be refined.

Corollary 1.5. *The restriction of β to a subset $B_1 \subseteq B$ such that no two elements of B_1 have a difference of* 1 *is injective. In particular, for $b \geq 3$ the restriction of β to $\{0, b-1\}^{\mathbb{N}}$ is injective.*

Proof. Follows from Lemma 1.4(b). □

1.2 Elements of Cantor sets

Proposition 1.6. *Let $x \in [0, 1]$. The following are equivalent:*

(i) $x \in \text{Cantor}_b$*;*

(ii) x can be written in base b as $x = \sum_{k \in \mathbb{N}} \frac{c_k}{b^k}$ where $c_k \in \{0, b-1\}$, that is $x = \beta(c)$ for some $c \in \{0, b-1\}^{\mathbb{N}}$.

Proof. Since an element $x \in {*}_b^n\, [0, 1]$ lies in in one of the intervals which makes up this set, it follows from the definition of ${*}_b^n [0, 1]$ that $l \leq x \leq l + b^{-n}$, where $l \in L_n^b$ (see Lemma 1.1(ii)). By Remark 1.3 and Lemma 1.1(iv) it follows that $x = \sum_{k \in \mathbb{N}} \frac{c_k}{b^k}$ where $c_i \in \{0, b-1\}$ for $1 \leq i \leq n$. The claimed equivalence follows because Cantor_b is the intersection of ${*}_b^n [0, 1]$. □

We temporarily work with ordinary base ten numbers ($b = 10$) in order to illustrate the construction. The elements of ${*}_{10}^1 [0, 1]$ are elements of $[0, \frac{1}{10}] \cup [\frac{9}{10}, 1]$, so in ordinary decimal form (positional notation base 10) have the form

$$0.0c_2c_3c_4 \ldots_{[10]} \quad \text{or} \quad 0.9c_2c_3c_4 \ldots_{[10]} .$$

The elements of ${*}_{10}^{2}[0,1]$ are elements of $[0,\frac{1}{10^2}]\cup[\frac{9}{10^2},\frac{1}{10}]\cup[\frac{9}{10},\frac{91}{10^2}]\cup[\frac{99}{10^2},1]$, so in ordinary decimal form (positional notation base 10) have the form

$$0.00c_3c_4\ldots_{[10]} \quad \text{or} \quad 0.09c_3c_4\ldots_{[10]} \quad \text{or} \quad 0.90c_3c_4\ldots_{[10]} \quad \text{or} \quad 0.99c_3c_4\ldots_{[10]}\,.$$

The elements of ${*}_{10}^{3}[0,1]$ have the form

$$0.000c_4\ldots_{[10]} \quad \text{or} \quad 0.009c_4\ldots_{[10]} \quad \text{or} \quad 0.090c_4\ldots_{[10]} \quad \text{or} \quad 0.099c_4\ldots_{[10]}$$
$$0.900c_4\ldots_{[10]} \quad \text{or} \quad 0.009c_4\ldots_{[10]} \quad \text{or} \quad 0.990c_4\ldots_{[10]} \quad \text{or} \quad 0.999c_4\ldots_{[10]}\,.$$

This pattern continues.

Corollary 1.7. *The Cantor set* Cantor_b *is uncountable.*

Proof. Since b is at least 3 Proposition 1.6 implies that every element of Cantor_b is equal to $\beta(c)$ where $c \in \{0, b-1\}^{\mathbb{N}}$, and by Corollary 1.5 β is injective when restricted to this set. Hence there is a bijective correspondence between Cantor_b and $\{0, b-1\}^{\mathbb{N}}$, so it suffices to see that the latter set is uncountable. This is a standard diagonal argument. Suppose there is a bijection between $\mathbb{N}$ and $\{0, b-1\}^{\mathbb{N}}$, and let $c^{(n)}$ denote the image of $n \in \mathbb{N}$ under this bijection. Define $d \in \{0, b-1\}^{\mathbb{N}}$ by requiring $d_k \neq c_k^{(k)}$ (so if $c_k^{(k)} = 0$ then $d_k = b-1$ and vice-versa). Then $d \neq c^{(n)}$ for all $n \in \mathbb{N}$ as $d_n \neq c_n^{(n)}$, which means that the supposed bijection cannot exist. □

Remark 1.8. *The above result is quite surprising, because it seems like we removed a lot of the interval* $[0,1]$ *in constructing* Cantor_b*. Indeed, taking* $b = 10$ *we see, using Lemma 1.1, that in constructing* ${*}_b^{n}[0,1]$ *from* ${*}_b^{n-1}[0,1]$ *we remove a total length of* $2^{n-1}\frac{8}{10^n}$*. Hence the total length removed in the construction of* Cantor_{10} *is*

$$\sum_{n=1}^{\infty} 2^{n-1}\frac{8}{10^n} = \sum_{n=1}^{\infty} \frac{2^n 4}{2^n 5^n} = \frac{4}{5}\sum_{n=0}^{\infty}\frac{1}{5^n} = \frac{4}{5}\left(\frac{1}{1-\frac{1}{5}}\right) = 1.$$

So after removing a total length equal to the length of the original interval $[0,1]$ *we are still left with uncountably many points.*

Remark 1.9. *It follows immediately from Proposition 1.6 and Lemma 1.1(iv) that the left and right endpoints of all intervals involved in the construction* Cantor_b *belong to* Cantor_b*. An example of one of the uncountably many other points in* Cantor_b *is*

$$\frac{1}{11} = 0.090909\ldots_{[10]}$$

belongs to Cantor$_{10}$*, but is not the endpoint of any intervals involved in the construction.*

Remark 1.10. *Proposition 1.6 tells us that each element of* Cantor$_b$ *can be identified with a sequence, where each term of the sequence is either* 0 *or* $b-1$ *(see the proof of Corollary 1.7). In other words, we have a bijection*

$$\text{Cantor}_b \longleftrightarrow \{0,1\}^{\mathbb{N}},$$

induced by β*. This bijection identifies a sequence* $a = (a_1, a_2, a_3, \ldots) \in \{0,1\}^{\mathbb{N}}$ *(so* $a_n \in \{0,1\}$*) with the base* b *number* $0.d_1d_2d_3\ldots_{[b]}$*, where the digits are* $d_i = (b-1)a_i$*, which belongs to* Cantor$_b$ *by Proposition 1.6. We will show this identification is correct from a topological point of view in Theorem 3.5.*

We close this section by recording some facts about arithmetic and Cantor sets. The first is that the average of any pair of elements of Cantor$_5$ does not belong to Cantor$_5$, but this is not true for Cantor$_3$, which means Cantor sets in different bases behave differently under certain arithmetic operations. Suppose $x, y \in \text{Cantor}_5$ are distinct points, so in the base 5 expansions $x = \sum_{k\in\mathbb{N}} c_k 5^{-k}$ and $y = \sum_{k\in\mathbb{N}} d_k 5^{-k}$ there is a smallest $n \in \mathbb{N}$ such that $c_k \neq d_k$. This means that the nth digit of the average $\frac{1}{2}(x+y)$ will be a 2, or a 3 if there is carrying in the addition, so the average does not belong to Cantor$_5$ by Proposition 1.6. On the other hand, 0 and 2/3 both belong to Cantor$_3$ (they are left endpoints from the first step) and their average is 1/3, which also belongs to Cantor$_3$ — it is a right endpoint of the first step.

Second, we claim that $\text{Cantor}_3 + \text{Cantor}_3 = [0,2]$. That $\text{Cantor}_3 + \text{Cantor}_3 \subseteq [0,2]$ is clear. To show the reverse inclusion we will show the equivalent statement that $[0,1] \subseteq \frac{1}{2}\text{Cantor}_3 + \frac{1}{2}\text{Cantor}_3$. Let $x \in [0,1]$ and write x in base 3 as

$$x = \sum_{k\in\mathbb{N}} \frac{x_k}{3^k}, \quad x_k \in \{0,1,2\}.$$

Now for $k \in \mathbb{N}$ define

$$y_k := \begin{cases} 0 & \text{if } x_k = 0, \\ 1 & \text{if } x_k \in \{1,2\}, \end{cases} \qquad \text{and} \qquad z_k := \begin{cases} 0 & \text{if } x_k \in \{0,1\}, \\ 1 & \text{if } x_k = 2. \end{cases}$$

Hence $y_k + z_k = x_k$ for each $k \in \mathbb{N}$. Also $y = \sum_{k\in\mathbb{N}} y_k 3^{-k} \in \frac{1}{2}\text{Cantor}_3$ because

$$2y = \sum_{k\in\mathbb{N}} \frac{2y_k}{3^k}, \quad 2y_k \in \{0,2\},$$

and similarly $z = \sum_{k\in\mathbb{N}} z_k 3^{-k} \in \frac{1}{2}\mathrm{Cantor}_3$. We have

$$y + z = \sum_{k\in\mathbb{N}} \frac{y_k}{3^k} + \sum_{k\in\mathbb{N}} \frac{z_k}{3^k} = \sum_{k\in\mathbb{N}} \frac{y_k + z_k}{3^k} = \sum_{k\in\mathbb{N}} \frac{x_k}{3^k} = x.$$

This completes the proof of the second fact.

The third arithmetic fact is that $*_b^n[0,1] = L_n^b + [0, \frac{1}{b^n}]$, which is immediate from the definitions. Indeed, an element $z \in *_b^n [0,1]$ must be in one of the intervals which makes up this set, hence can be written as $x + y$, where $x \in L_n^b$ is the left endpoint of the interval containing z and $y \in [0, \frac{1}{b^n}]$ by Lemma 1.1(ii).

2 Topological properties

The Cantor sets Cantor_b are topological spaces in the subspace topology from $\mathbb{R}$ (or from $[0,1]$). Now we investigate topological properties of Cantor sets. The first lemma is immediate from the topological properties of $\mathbb{R}$, we state it for the record.

Lemma 2.1. *The Cantor sets* Cantor_b *are metrisable and separable.*

Proposition 2.2. *The Cantor sets* Cantor_b *are closed.*

Proof. By construction each set $*_b^n[0,1]$ is a finite union of closed intervals, therefore is closed. Hence Cantor_b is closed as the intersection of a family of closed sets. □

Corollary 2.3. *The Cantor sets* Cantor_b *are compact.*

Proof. The Cantor sets are closed subsets of the compact set $[0,1]$, hence compact. □

We now give some more information about neighbourhoods in Cantor sets.

Proposition 2.4. *(i) Every neighbourhood of a point* $x \in \mathrm{Cantor}_b$ *contains a set of the form* $\mathrm{Cantor}_b \cap [l, l + b^{-n}]$*, where* $l \in \mathrm{Cantor}_b$ *is a left endpoint of an interval from the construction of* Cantor_b *and* $n \in \mathbb{N}$.

(ii) The sets of the form in (i) form a basis for the topology on Cantor_b *which consists of closed-open sets.*

Proof. (i) Let N be a neighbourhood of x in the topology of Cantor_b. Using the definition of the subspace topology of Cantor_b inherited from $\mathbb{R}$ there is $\epsilon > 0$ such that $\mathrm{Cantor}_b \cap (x-\epsilon, x+\epsilon) \subseteq N$. Choosing $n \in \mathbb{N}$ such that $b^{-n} < \epsilon$ then, since $x \in^{*n}_{b} [0,1]$, there is a left end-point $l \in L^b_n$ with $x \in [l, l+b^{-n}]$ (see Lemma 1.1). By our choice of n we have $[l, l+b^{-n}] \subseteq (x-\epsilon, x+\epsilon)$, so $\mathrm{Cantor}_b \cap [l, l+b^{-n}] \subseteq N$.

(ii) We first show the given sets are closed and open. It follows from Lemma 1.1 that if $l \in L^b_n$ is a left endpoint of one of the intervals in the construction of Cantor_b then for $n \geq 1$ we have

$$\mathrm{Cantor}_b \cap [l, l+b^{-n}] = \mathrm{Cantor}_b \cap (l - b^{-n}, l + 2b^{-n}).$$

This is obviously both closed and open in the topology of Cantor_b, which is the subspace topology inherited from $\mathbb{R}$. To see that the sets form a basis suppose $U \subseteq \mathrm{Cantor}_b$ is open. Then, by (i), for each $x \in U$ there is a set $B_x = \mathrm{Cantor}_b \cap [l, l+b^{-n}]$ such that $x \in B_x \subseteq U$. Hence

$$U \subseteq \cup_{x\in U} B_x \subseteq U,$$

so $U = \cup_{x\in U} B_x$. This shows that sets of the form $\mathrm{Cantor}_b \cap [l, l+b^{-n}]$ form a basis for the topology of Cantor_b. □

An *isolated point* of a subset S of a topological space X is a point $x \in S$ which has a neighbourhood containing no points of S except for x; equivalently $\{x\}$ is an open set in the subspace topology of S.

Proposition 2.5. *The Cantor set* Cantor_b *has no isolated points.*

Proof. Take $x \in \mathrm{Cantor}_b$. By definition of the subspace topology it suffices to show that for any $\epsilon > 0$ there exists $y \in \mathrm{Cantor}_b \cap (x-\epsilon, x+\epsilon)$. Choose $n \in \mathbb{N}$ such that $1/b^n < \epsilon$. Suppose $x = \sum_{k\in\mathbb{N}} c_k b^{-k}$ where $c_k \in \{0, b-1\}$ and define $y \in \mathrm{Cantor}_b$ by

$$y = \sum_{k\in\mathbb{N}} d_k b^{-k}, \quad \text{where} \quad d_k := \begin{cases} c_k & \text{if } k \neq n \\ (b-1) - c_k & \text{if } k = n. \end{cases}$$

(So the base b expansion of y is the same as that of x except for the nth digit, which is the element of $\{0, b-1\} \setminus \{c_n\}$.) Then $x \neq y$ and $y \in \mathrm{Cantor}_b$ by Proposition 1.6, while Remark 1.3 tells us that $|x-y| < 1/b^n < \epsilon$ as required. □

A subset A of a topological space X is called *perfect* if A is closed and has no isolated points; equivalently each point of A is an accumulation point of A, or A is dense in itself.

Corollary 2.6. *The Cantor sets* Cantor_b *are perfect.*

A set is called *nowhere dense* if its closure has empty interior.

Proposition 2.7. *The Cantor sets* Cantor_b *are nowhere dense.*

Proof. It suffices to show that Cantor_b does not contain a non-empty open interval. Suppose $c, d \in [0,1]$ with $c < d$ and choose $n \in \mathbb{N}$ with $b^{-n} < d - c$. By Lemma 1.1 (c, d) is not contained in $*_b^n[0,1]$, so is not contained in Cantor_b. □

The above result can also be deduced from the computation of the "length" of Cantor_b in Remark 1.8: this computation (can be formalised so it) shows that the *Lebesgue measure* of Cantor_b is 0, which is incompatible with Cantor_b containing a non-empty open interval.

Recall that a subset Y of a topological space is *disconnected* if it can be written as $Y = A \cup B$, where A and B are nonempty disjoint open (equivalently closed) sets. A topological space is called *totally disconnected* if the only nonempty connected subsets are singletons. A topological space X is called *0-dimensional*[1] if each point has a neighbourhood base consisting of open-closed sets; equivalently, for every $x \in X$ and closed set A not containing x there is an open-closed set which contains x and does not meet A. Since $\{x\}$ is closed for each $x \in \mathrm{Cantor}_b$ it is clear that 0-dimensional is a stronger property than total disconnectedness.

Proposition 2.8. *The Cantor sets* Cantor_b *are 0-dimensional, hence totally disconnected.*

Proof. Proposition 2.4 shows that every point of Cantor_b has a neighbourhood base consisting of open-closed sets. □

3 Homeomorphisms

Finally, we prove the existence of some homeomorphisms involving Cantor sets. The first shows that Cantor sets behave like a fractal.

Proposition 3.1. *Let* $n \in \mathbb{N}$ *and* $l \in L_n^b$ *a left endpoint of one of the intervals* $[l, l + b^{-n}]$ *which make up* $*_b^n[0,1]$. *The intersection* $\mathrm{Cantor}_b \cap [l, l + b^{-n}]$ *is homeomorphic to* Cantor_b.

[1]This definition is taken from [Wil, Definition 29.4]. There are other notions of zero-dimensional topological space, which may not be equivalent to the one here in general. However, since Cantor sets are separable and metrisable the notions coincide.

Proof. Consider the map

$$[l, l+b^{-n}] \to [0, b^{-n}];\ x \mapsto x - l.$$

This map is clearly a homeomorphism. Also, since l has the form stated in Lemma 1.1, we see from Remark 1.3 that the given map does not change any base b digit after the nth, while the first n base b digits of l are the same as the first n base b digits of every element of $[l, l+b^{-n}]$. It follows that the given map restricts to a homeomorphism

$$\mathrm{Cantor}_b \cap [l, l+b^{-n}] \to \mathrm{Cantor}_b \cap [0, b^{-n}];\ x \mapsto x - l.$$

Now consider the map

$$\mathrm{Cantor}_b \cap [0, b^{-n}] \to \mathrm{Cantor}_b;\ y \mapsto b^n y,$$

which simply 'removes' the first n base b digits of y (all of which are 0). It is clear that this map is a homeomorphism between the stated sets. The composition of these two homeomorphisms gives the statement. □

The following result means that, from a topological point of view, there is no problem with referring to *the Cantor set*.

Proposition 3.2. *For any two bases b_1 and b_2 ($b_1, b_2 \in \mathbb{N}$ and $b_1, b_2 \geq 3$) the Cantor sets Cantor_{b_1} and Cantor_{b_2} are homeomorphic.*

Proof. Consider the natural map

$$\theta : \mathrm{Cantor}_{b_1} \to \mathrm{Cantor}_{b_2};\ \theta\left(\sum_{k\in\mathbb{N}} c_k b_1^{-k}\right) := \sum_{k\in\mathbb{N}} \left(\frac{b_2 - 1}{b_1 - 1} c_k\right) b_2^{-k}.$$

Since $c_k \in \{0, b_1 - 1\}$ we have $\frac{b_2-1}{b_1-1} c_k \in \{0, b_2 - 1\}$, so the definition makes sense. It is immediate from Proposition 1.6 that θ is bijective. To show continuity of θ take $\epsilon > 0$. Find $n \in \mathbb{N}$ such that $b_2^{-n} < \epsilon$, then let $\delta = b_1^{-n}$. If $x, y \in \mathrm{Cantor}_{b_1}$ and $|x - y| < \delta$ then, writing c_k and d_k for the base b_1 digits of x and y respectively, we must have $c_k = d_k$ for all $k \leq n$ (see

Remark 1.3), so

$$
\begin{aligned}
\left|\theta(x)-\theta(y)\right| &= \left|\sum_{k\in\mathbb{N}}\left(\frac{b_2-1}{b_1-1}c_k\right)b_2^{-k}-\sum_{k\in\mathbb{N}}\left(\frac{b_2-1}{b_1-1}d_k\right)b_2^{-k}\right| \\
&= \left|\sum_{k\in\mathbb{N}}\frac{b_2-1}{b_1-1}b_2^{-k}(c_k-d_k)\right| \\
&= \frac{b_2-1}{b_1-1}\left|\sum_{k\in\mathbb{N}}b_2^{-k}(c_k-d_k)\right| \\
&< \frac{b_2-1}{b_1-1}b_2^{-n} < \frac{b_2-1}{b_1-1}\epsilon,
\end{aligned}
$$

where the penultimate inequality follows from Remark 1.3. This proves that θ is continuous. The same argument proves continuity of θ^{-1}, as

$$
\theta^{-1}\left(\sum_{k\in\mathbb{N}}c_k b_2^{-k}\right)=\sum_{k\in\mathbb{N}}\left(\frac{b_1-1}{b_2-1}c_k\right)b_1^{-k}
$$

(the roles of b_1 and b_2 are exchanged). Thus θ is a homeomorphism. (In the proof of continuity of θ we chose x and y in the same interval of ${*}_{b_1}^{n}[0,1]$, which ensures $\theta(x)$ and $\theta(y)$ are in the same interval of ${*}_{b_1}^{n}[0,1]$, therefore the distance between $\theta(x)$ and $\theta(y)$ is at most b_2^{-n}.) $\square$

In fact it is possible to prove much more than the above result; the proof of the following result, sometimes called Brouwer's Theorem, can be found in [Wil, Theorem 30.3].

Theorem 3.3. *Any two totally disconnected, perfect, compact metric spaces are homeomorphic.* $\square$

The following immediate consequence is due to Hausdorff [Hau].

Corollary 3.4. *The Cantor set is, up to homeomorphism, the unique totally disconnected, perfect, compact metric space.*

In Remark 1.10 we saw that Cantor sets can be written as an infinite product. We now show that this bijection is a homeomorphism.

Theorem 3.5. *The Cantor set is homeomorphic to the product space* $\{0,1\}^{\mathbb{N}}$.

Proof. Consider the map induced by β, mentioned in Remark 1.10

$$\tilde{\beta} : \{0,1\}^{\mathbb{N}} \to \text{Cantor}_b;\ (a_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} \frac{a_k(b-1)}{b^k}.$$

Since $\{0,1\}^{\mathbb{N}}$ is compact and Cantor_b is Hausdorff a standard topological result (see for example [Wil, Theorem 17.14]) says that to prove $\tilde{\beta}$ is a homeomorphism we only need to prove it is continuous. Choose $x \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$. We want to find an open neighbourhood N of x such that if $y \in N$ then $|\tilde{\beta}(x) - \tilde{\beta}(y)| < b^{-n}$; this will show $\tilde{\beta}$ is continuous at x, hence it is continuous as x is arbitrary. Define $N := \cap_{i=1}^{n} \pi_i^{-1}(\{x_i\})$, which is an open set in the product space $\{0,1\}^{\mathbb{N}}$. (Here $\pi_i : \{0,1\}^{\mathbb{N}} \to \{0,1\};\ (x_k)_{k\in\mathbb{N}} \mapsto x_i$ is the canonical projection.) Then $y \in N$ means $x_i = y_i$ for $1 \le i \le n$, therefore $|\tilde{\beta}(x) - \tilde{\beta}(y)| < b^{-n}$ by Remark 1.3. This shows that $\tilde{\beta}$ is continuous, hence a homeomorphism. □

Corollary 3.6. *The Cantor set is (homeomorphic to) an abelian compact topological group.*

Proof. The set $\{0,1\} = \mathbb{Z}_2$ is an abelian compact topological group, therefore so is the Cantor set by Theorem 3.5. □

A topological space X is called *homogeneous* if for any two points $x, y \in X$ there is a homeomorphism on X which maps x to y.

Proposition 3.7. *The Cantor set is homogeneous.*

Proof. For any topological group G and any $g \in G$ the map

$$G \to G;\ h \mapsto gh$$

is a homeomorphism (see [Wil, Section 13.G]). In particular, by Corollary 3.6, for any pair of elements x, y of the Cantor set there is a homeomorphism $z \mapsto yx^{-1}z$. This homeomorphism clearly maps x to y. □

Note that the above result shows that there is nothing topologically special about the endpoints of the intervals involved in the construction of the Cantor set.

To finish we quote a result of Alexandroff and Urysohn which gives a universal property of the Cantor set (see [Wil, Theorem 30.7]).

Theorem 3.8. *Every compact metric space is a continuous image of the Cantor set.* □

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