MATH 116 REVISION

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Part 1: Integrals

Sums. Sigma notation is a compact way to write the sum of several terms:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n.$$

The main properties of this notation are:

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \quad \sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i, \quad \sum_{i=1}^{n} c = cn.$$

Some sums are *telescoping*, which means that most of their terms cancel out.

Example. *Calculate* $\displaystyle\sum_{i=3}^{50}\left(\frac{1}{5^i} - \frac{1}{5^{i+1}}\right)$.

Solution. We have

$$\begin{aligned}\sum_{i=3}^{50}\left(\frac{1}{5^i} - \frac{1}{5^{i+1}}\right) &= \left(\frac{1}{5^3} - \frac{1}{5^4}\right) + \left(\frac{1}{5^4} - \frac{1}{5^5}\right) + \cdots + \left(\frac{1}{5^{49}} - \frac{1}{5^{50}}\right) + \left(\frac{1}{5^{50}} - \frac{1}{5^{51}}\right)\\ &= \frac{1}{5^3} - \frac{1}{5^{51}} = \frac{1}{5^3}\left(1 - \frac{1}{5^{48}}\right).\end{aligned}$$

□

There are useful formulae for sums such as $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, which will be provided in exams.

Area. The area of a rectangle is the product of its length and width; the area of a triangle is $\frac{1}{2}bh$, where b is the length of the base and h the perpendicular height. If a shape cannot be split into rectangles and triangles, for example if the edges are curved, then it is not so easy to define its area.

To calculate the area of any region we first approximate the area by dividing the region into n rectangles of equal width, the length of each rectangle varies according to the shape of the region, and the area of the region is approximately the sum of the areas of the rectangles (which we know how to calculate). Letting the number of rectangles n go to infinity makes the error of the approximation smaller, so the area should be the limit of the areas of the rectangles.

If f is a continuous function on the interval $[a, b]$ then the area A under f between a and b (*i.e.* the area between f, the x-axis and the lines $x = a$ and $x = b$) is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

This is the limit of the right endpoint approximations R_n; equivalently, A can be expressed as the limit of the left endpoint approximations L_n:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

If $v(t)$ is the velocity function of an object then the distance travelled by the object in a time interval can be approximated by areas of rectangles similarly to above; in this case the width of an approximating rectangle is a small time interval Δt, the height of this rectangle is the velocity of the particle at some point during the small time interval, so the area of the rectangle is the distance the object travels during that time interval if the velocity is constant over that time interval.

The definite integral. The definite integral $\int_a^b f(x)\,dx$ of a function f over an interval $[a,b]$ represents a difference of areas on the interval $[a,b]$: the area above the x-axis and under f subtract the area under the x-axis and above f. The numbers a and b are called the *lower and upper limits of integration*; the function $f(x)$ is the *integrand.*

Definition. *If f is a function defined on $[a,b]$ divide the interval $[a,b]$ in n subintervals of width $\Delta x = \frac{b-a}{n}$, with endpoints $x_0 = a, x_1, x_2, \dots, x_{n-1}, x_n = b$ where $x_i = a + i\Delta x$. Let $x_i^* \in [x_{i-1}, x_i]$. The* definite integral *of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided this limit exists and has the same value for any choice of the numbers x_i^*. We often choose $x_i^* = x_i$ and use

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Theorem. *If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities on $[a,b]$, then f is integrable on $[a,b]$.*

We calculate definite integrals using the properties of limits and the summation formulae.

Example. *Calculate* $\displaystyle\int_{-1}^{2} x^2 - 2\,dx.$

Solution. We have $\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$, so $x_i = \frac{3i}{n} - 1$. Therefore, using the summation formulae,

$$\begin{aligned}
\int_{-1}^{2} x^2 - 2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n} - 1\right)^2 - 2\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} \frac{9i^2}{n^2} - \sum_{i=1}^{n} \frac{6i}{n} - \sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left(\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{18}{n^2}\frac{n(n+1)}{2} - \frac{3}{n}n\right) \\
&= \lim_{n\to\infty} \left(\frac{9}{2}\frac{n}{n}\frac{n+1}{n}\frac{2n+1}{n} - 9\frac{n}{n}\frac{n+1}{n} - 3\frac{n}{n}\right) \\
&= \frac{9}{2}(1)(1+0)(2+0) - 9(1)(1+0) - 3(1) = -3.
\end{aligned}$$

□

The definite integral has the following properties:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \quad \int_a^a f(x)\,dx = 0, \quad \int_a^b c\,dx = c(b-a), \quad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx, \quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

$$\int_a^b f(x)\,dx \geq 0 \text{ if } f(x) \geq 0 \text{ for all } x \in [a,b], \quad \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \text{ if } f(x) \geq g(x) \text{ for all } x \in [a,b],$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \text{ if } m \leq f(x) \leq M \text{ for all } x \in [a,b].$$

Fundamental theorem of calculus. Let f be continuous on $[a, b]$; we consider the function $g(x) = \int_a^x f(t)\,dt$. The function g represents "area so far" under f.

Theorem. *Suppose f is continuous on $[a, b]$.*

> **FTC1:** *If $g(x) = \int_a^x f(t)\,dt$ then g is continuous on $[a, b]$, differentiable on (a, b) and $g'(x) = f(x)$.*
> **FTC2:** *$\int_a^b f(x)\,dx = F(b) - F(a)$ for any antiderivative F of f ($F' = f$).*

FTC1, together with the properties of the definite integral, can be used to calculate certain derivatives.

Example. *Find the derivative of $g(x) = \displaystyle\int_{\sqrt{x}}^{\pi/4} \theta \tan(\theta)\,d\theta$.*

Solution. Using the properties of the definite integral

$$g(x) = -\int_{\pi/4}^{\sqrt{x}} \theta \tan(\theta)\,d\theta = -f \circ h(x),$$

where $h(x) = \sqrt{x}$ and $f(x) = \int_{\pi/4}^x \theta \tan(\theta)\,d\theta$. By the chain rule $g'(x) = -f'(h(x))h'(x)$. We have $h'(x) = \frac{1}{2\sqrt{x}}$ and by FTC1 $f'(x) = x\tan(x)$, so

$$g'(x) = -f'(h(x))h'(x) = -\sqrt{x}\tan\left(\sqrt{x}\right)\frac{1}{2\sqrt{x}} = -\frac{1}{2}\tan\left(\sqrt{x}\right).$$

□

FTC2 allows us to calculate definite integrals using antiderivatives:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is any antiderivative for f.

Example. *Calculate $\displaystyle\int_0^3 2\sin(x) - e^x\,dx$.*

Solution. Using FTC2

$$\begin{aligned}\int_0^3 2\sin(x) - e^x\,dx = [-2\cos(x) - e^x]_0^3 &= \left(-2\cos(3) - e^3\right) - \left(-2\cos(0) - e^0\right) \\ &= 3 - 2\cos(3) - e^3,\end{aligned}$$

since $\frac{d}{dx}(-2\cos(x) - e^x) = 2\sin(x) - e^x$. □

The fundamental theorem of calculus tells us that differentiation and integration are inverse processes. FTC1 says that $\frac{d}{dx}(\int_a^x f(t)\,dt) = f(x)$, *i.e.* the derivative of the integral of f is f. FTC2 says that $\int_a^b f'(x)\,dx = f(b) - f(a)$, *i.e.* the integral of the derivative of f can be found from f.

Indefinite integrals. FTC2 says that definite integrals can be calculated using antiderivatives, so we write antiderivatives using integral notation: $\int f(x)\,dx$ is the *indefinite integral* of f and $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$. An indefinite integral represents a family of functions (remember a definite integral represents a number), so we always write $\int f(x)\,dx = F(x) + c$, where $F' = f$ and c represents a constant. We write indefinite integrals in this way even though the expression may only be valid on an interval, not on all of the real line.

FTC2 says that $\displaystyle\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$. The example $\int_0^3 2\sin(x) - e^x\,dx$ above can be seen as an example of calculating a definite integral using indefinite integrals.

Net change. The net change of a function f over an interval $[a, b]$ is the difference $f(b) - f(a)$. The net change theorem is a consequence of FTC2.

Theorem. *The integral of a rate of change is the net change, i.e.* $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Examples of net changes include:

- if $P(t)$ represents the population of a country at time t (which may increase at some times and decrease at others) the rate of growth of the population is $\frac{dP}{dt}$, and from time t_1 to t_2 the overall change in population is $\int_{t_1}^{t_2} \frac{dP}{dt}\,dt = P(t_2) - P(t_1)$.
- If $V(t)$ represents the volume of water in a container at time t then $\frac{dV}{dt}$ is the change in volume (which may be positive or negative depending on how much water is flowing in and out) and from time t_1 to t_2 the overall change in volume is $\int_{t_1}^{t_2} \frac{dV}{dt}\,dt = V(t_2) - V(t_1)$.
- If $v(t)$ represents the velocity of an object at time t, so $v(t) = s'(t)$ with s the position function of the object, then from time t_1 to t_2 the displacement is $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$. The total distance travelled by the object in this time interval is $\int_{t_1}^{t_2} |v(t)|\,dt$ (a technique for calculating this type of integral is given below).

Basic integration techniques. To find the definite integral of $\int_a^b |f(x)|\,dx$ split the interval $[a, b]$ in subintervals such that f does not change sign on each subinterval, then use that $|y| = y$ if $y \geq 0$ and $|y| = -y$ if $y < 0$.

Example. *Calculate* $\int_0^2 \left|t^3 - 1\right|\,dt$.

Solution. Note that $t^3 - 1 < 0 \iff t < 1$ and $t^3 - 1 \geq 0 \iff t \geq 1$, so we split the interval $[0, 2]$ in subintervals $[0, 1]$ and $[1, 2]$. Then

$$\begin{aligned}\int_0^2 \left|t^3 - 1\right|\,dt &= \int_0^1 \left|t^3 - 1\right|\,dt + \int_1^2 \left|t^3 - 1\right|\,dt = \int_0^1 1 - t^3\,dt + \int_1^2 t^3 - 1\,dt \\ &= \left[t - \frac{t^4}{4}\right]_0^1 + \left[\frac{t^4}{4} - t\right]_1^2 = 1 - \frac{1}{4} + 2 - \frac{3}{4} = 2.\end{aligned}$$

□

Integration by substitution is the integral version of the chain rule; it allows us to calculate integrals of the form $\int f'(g(x))g'(x)\,dx$ by choosing $u = g(x)$, so $du = g'(x)\,dx$ and $\int f'(g(x))g'(x)\,dx = \int f'(u)\,du$. It may happen that one needs to multiply by a constant so that $cdu = g'(x)\,dx$ as in the second example below; it is helpful to remember that one can work with du and dx as if they obey the normal rules of algebra.

Example. *Calculate* $\int \cot(\theta)\,d\theta$.

Solution. Recall $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$. Let $u = \sin(\theta)$, so $du = \cos(\theta)\,d\theta$. Therefore

$$\int \cot(\theta)\,d\theta = \int \frac{\cos(\theta)}{\sin(\theta)}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + c = \ln|\sin(\theta)| + c.$$

□

When finding a definite integral by substitution remember to keep track of how the limits of integration change (or change back to the original variable before evaluating at the limits of integration).

Example. *Calculate* $\int_0^{\frac{\pi}{4}} 3^{2\sec(x)} \sec(x)\tan(x)\,dx$.

Solution. Let $u = 2\sec(x)$, so $\frac{du}{dx} = 2\sec(x)\tan(x)$. Thus $du = 2\sec(x)\tan(x)\,dx$, so $\frac{1}{2}du = \sec(x)\tan(x)\,dx$. We have $x = 0 \implies u = 2$ and $x = \pi/4 \implies u = 4/\sqrt{2} = 2\sqrt{2}$, so

$$\int_0^{\frac{\pi}{4}} 3^{2\sec(x)} \sec(x)\tan(x)\,dx = \int_2^{2\sqrt{2}} \frac{1}{2}3^u\,du = \frac{1}{2}\left[\frac{3^u}{\ln(3)}\right]_2^{2\sqrt{2}} = \frac{1}{2\ln(3)}\left(3^{2\sqrt{2}} - 3^2\right).$$

□

Some choices of substitution will not make the integral easier to solve (for example $u = \sec(x)\tan(x)$ in the example above), but there may be several substitutions that work.

Alternative solution. Let $u = 3^{2\sec(x)}$, so $du = 3^{2\sec(x)} 2\ln(3)\sec(x)\tan(x)\,dx$, so rearranging gives $\frac{1}{2\ln(3)}du = 3^{2\sec(x)}\sec(x)\tan(x)\,dx$. Thus

$$\int_0^{\frac{\pi}{4}} 3^{2\sec(x)}\sec(x)\tan(x)\,dx = \int_{x=0}^{x=\frac{\pi}{4}} \frac{1}{2\ln(3)}\,du = \frac{1}{2\ln(3)}\,[u]_{x=0}^{x=\frac{\pi}{4}} = \frac{1}{2\ln(3)}\left[3^{2\sec(x)}\right]_0^{\frac{\pi}{4}} = \frac{1}{2\ln(3)}\left(3^{2\sqrt{2}} - 3^2\right).$$

□

Substitutions can be also be used to simplify the integrand.

Example. *Calculate* $\int \sqrt{1+x^4}x^7\,dx$.

Proof. Substitute $u = 1 + x^4$, so $du = 4x^3\,dx \implies \frac{1}{4}du = x^3\,dx$. The factor x^7 in the integrand becomes $\frac{1}{4}x^4\,du$, and $x^4 = u - 1$. Therefore

$$\int \sqrt{1+x^4}x^7\,dx = \int \frac{1}{4}\sqrt{u}(u-1)\,du = \frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c.$$

□

Tips for integration by substitution:

- look for a factor $f(u)$ in the integrand, with u a function of the variable of integration and an expression like $\frac{du}{dx}$ also a factor in the integrand;
- practice and don't be discouraged: if a substitution does not work try another.

Part 2: Applications of integration

Area between curves. To find the area enclosed by two curves we approximate the area using rectangles, then take the limit of the total area of these rectangles, just as we did when defining the definite integral as the area between a curve and the x-axis.

Definition. *Let f and g be continuous functions and $f(x) \geq g(x)$ for $a \leq x \leq b$. The area of the region enclosed by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$A = \int_a^b f(x) - g(x)\,dx.$$

To answer questions on area between curves you will sometimes have to decide:

- to integrate with respect to x or y;
- the limits of integration a and b;
- which curve is above and which below, and if this is different on subintervals of $[a, b]$;
- which curves bound the region if more than two curves are given.

The general formula for the area between curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx;$$

deciding which curve is above and below on a subinterval amounts to our technique for integrating absolute values by splitting up the integral. The steps are:

(1) decide if integration should be with respect to x (curves given as $y = f(x)$ or with respect to y (curves given as $x = g(y)$);
(2) find all intersection points, remembering any restrictions in the question: the least and greatest of these are the limits of integration (unless another line is specified), the others are when the curves cross;
(3) draw a diagram with everything you know;

(4) using the intersection points split the interval in subintervals, use test values to determine which curve is above and below on each interval and check this matches the diagram;
(5) write the area as a sum of integrals according to what you have found, find the area by evaluating the integrals;
(6) remember that evaluating the integrals may require a substitution, or some of the other techniques explained below.

Example. *(a) Find the area enclosed by the curves $x = y^3$ and $x = y$.*
(b) Find the area enclosed by the curves $y = \frac{1}{4}x^2$, $y = 2x^2$ and $x + y = 3$ in the region $x \geq 0$.

Solution. (a) Since one of the equations is of the form $x = g(y)$ we will integrate with respect to y. To find intersection points: $y^3 = y \iff y(y^2 - 1) = 0 \iff y = -1, 0, 1$; therefore our limits of integration are -1 and 1 and the curves cross when $y = 0$. The required area is $\int_{-1}^{1} |y^3 - y|\, dy$; to simplify the absolute value we split the integral over the intervals $[-1, 0]$ and $[0, 1]$. To see which curve is above on $[-1, 0]$ take $-1/2 \in [-1, 0]$: $(-1/2)^3 = -1/8 > -1/2$, so $x = y^3$ is above $x = y$ here. To see which curve is above on $[0, 1]$ take $1/2 \in [0, 1]$: $(1/2)^3 = 1/8 < 1/2$, so $x = y$ is above $x = y^3$ here. Hence the required area is

$$\int_{-1}^{1} |y^3-y|\, dy = \int_{-1}^{0} y^3-y\, dy+\int_{0}^{1} y-y^3\, dy = \left[\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{0} + \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_{0}^{1} = 0-\left(\frac{1}{4} - \frac{1}{2}\right)+\left(\frac{1}{2} - \frac{1}{4}\right)-0 = \frac{1}{2}.$$

(b) We will integrate with respect to x. Intersection points: $\frac{1}{4}x^2 = 2x^2 \iff x = 0$; $\frac{1}{4}x^2 = 3 - x \iff x^2 + 4x - 12 = 0 \iff x = -6, 2$; $2x^2 = 3 - x \iff 2x^2 + x - 3 = 0 \iff x = -3/2, 1$. The limits of integration are 0 and 2 ($x = -6$ and $x = -3/3$ are excluded by the condition $x \geq 0$); we will split our integral over the intervals $[0, 1]$ and $[1, 2]$. To see which curves are needed on $[0, 1]$ test with $x = 1/2$: $\frac{1}{4}(1/2)^2 = 1/16$, $2(1/2)^2 = 1/2$ and $3 - 1/2 = 5/2$, so $\frac{1}{4}x^2 \leq 2x^2 \leq 3 - x$ on this interval. On the interval $[1, 2]$ we find $\frac{1}{4}x^2 \leq 3 - x \leq 2x^2$. This shows the lower bound of the region is always the curve $y = \frac{1}{4}x^2$, and the upper bound of the region is the closer curve on each subinterval: $y = 2x^2$ on $[0, 1]$ and $y = 3 - x$ on $[1, 2]$. The required area is therefore

$$\int_{0}^{1} 2x^2-\frac{1}{4}x^2\, dx+\int_{1}^{2} (3-x)-\frac{1}{4}x^2\, dx = \left[\frac{7}{12}x^3\right]_{0}^{1}+\left[3x - \frac{x^2}{2} - \frac{x^3}{12}\right]_{1}^{2} = \frac{7}{12}+\left(6 - 2 - \frac{2}{3}\right)-\left(3 - \frac{1}{2} - \frac{1}{12}\right) = \frac{3}{2}.$$

□

Volumes. A *generalised cylinder* is a solid which has a face of area A and length l, slices of the object parallel to the face are the same shape and area as the face; such an object has volume $V = Al$. For example, a rectangular box or a normal cylinder with circular face.

To compute the volume of any solid we slice it in n pieces (which results in slicing the x-axis in n subintervals $[x_{i-1}, x_i]$ of width Δx), each piece is approximately a generalised cylinder of area $A(x_i)$ and thickness Δx, so with volume $A(x_i)\Delta x$. Adding the volumes of these generalised cylinders approximates the total volume of the solid, and the approximation improves as n becomes larger. Taking the limit we arrive at the definition of volume of a solid.

Definition. *Let S be a solid that lies between $x = a$ and $x = b$. The cross-sectional area of S, perpendicular to the x-axis, is $A(x)$ for some continuous function A. The volume of S is*

$$V = \int_{a}^{b} A(x)\, dx.$$

Example. *Find the volume of the solid S with base a circle of radius 1 and cross sections of S perpendicular to the base are squares.*

Solution. We choose the base to be the circle $x^2 + y^2 = 1$, which is centred at the origin. Since turning the base through $\pi/2$ radians gives the same shape we are free to choose any orientations for the slices whose integral gives the volume of S; we choose slices perpendicular to the x-axis so our integration is with respect to x. The slice of S at a distance x from the origin has base length $2y = 2\sqrt{1 - x^2}$, so the area of this slice is $A(x) = (2y)^2 = 4(1 - x^2)$. Since S lies between $x = -1$ and $x = 1$ the volume is

$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} 4(1 - x^2)\, dx = 8\int_{0}^{1} 1 - x^2\, dx = 8\left[x - \frac{x^3}{3}\right]_{0}^{1} = \frac{16}{3}.$$

(We used that the integrand is even to simplify the integral.) □

A *solid of revolution* is a solid formed by rotating a region around an axis.

The disc/washer method for finding the volume of a solid of revolution around a line of the form $y = c$, which is parallel to the x-axis ($y = 0$ is the x-axis) takes slices perpendicular to the x-axis with area $A(x) = \pi(r_2^2 - r_1^2)$, where r_2 is the outer radius of the washer and r_1 the inner radius. If the solid lies between $x = a$ and $x = b$ its volume is

$$V = \int_a^b A(x)\,dx = \int_a^b \pi(r_2^2 - r_1^2)\,dx;$$

The bounds $x = a$ and $x = b$ may be given, or may need to be worked out from the intersection points of the curves. To find the inner and outer radius:

(1) write the axis of rotation as $y = c$;
(2) write the curves which bound the region as $y = f(x)$ and $y = g(x)$; determine which of $|f(x) - c|$ or $|g(x) - c|$ is greater on the interval $[a, b]$;
(3) if $|f(x) - c|$ is greater then take $r_2 = f(x) - c$ and $r_1 = g(x) - c$, if $|g(x) - c|$ is greater then take $r_2 = g(x) - c$ and $r_1 = f(x) - c$ (strictly speaking we may have found $-r_1$ and $-r_2$ but it does not matter).

If the axis of rotation is a line $x = c$ ($x = 0$ is the y-axis) then integrate with respect to y, follow the above procedure with the roles of x and y exchanged.

Example. *Find the volume of the solid obtained by rotating the region bounded by $y = \sin(x)$, $y = \cos(x)$, $0 \le x \le \pi/4$, about the axis $y = -1$.*

Solution. Since the axis $y = -1$ is parallel to the x-axis we will integrate with respect to x. To find the inner and outer radius of a washer note $|\cos(x) - (-1)| = \cos(x) + 1$ and $|\sin(x) - (-1)| = \sin(x) + 1$ on the interval $[0, \pi/4]$; since $\cos(x) \ge \sin(x)$ on this interval we have $|\cos(x) - (-1)| \ge |\sin(x) - (-1)|$ on this interval, so the outer radius of a washer is $\cos(x) - (-1)$ and the inner radius is $\sin(x) - (-1)$. Hence the area of a washer is

$$\begin{aligned} A(x) = \pi\left(\cos(x) - (-1)\right)^2 - \pi\left(\sin(x) - (-1)\right)^2 &= \pi\left(\cos^2(x) + 2\cos(x) - \sin^2(x) - 2\sin(x)\right) \\ &= \pi\left(\cos(2x) + 2\cos(x) - 2\sin(x)\right), \end{aligned}$$

the last equality uses a trig identity to make the integration below easier. Therefore the volume of the solid is

$$\begin{aligned} V = \int_a^b A(x)\,dx = \int_0^{\frac{\pi}{4}} \pi\left(\cos(2x) + 2\cos(x) - 2\sin(x)\right)\,dx &= \pi\left[\frac{1}{2}\sin(x) + 2\sin(x) + 2\cos(x)\right]_0^{\frac{\pi}{4}} \\ = \pi\left(\left(\frac{1}{2} + \sqrt{2} + \sqrt{2}\right) - (0 + 0 + 2)\right) &= \pi\left(2\sqrt{2} - \frac{3}{2}\right). \end{aligned}$$

□

In certain cases it can be difficult, or even impossible, to calculate the volume of a solid of revolution by the disc/washer method, because it is difficult to write the (inner or outer) radius above in terms of the variable of integration. When this happens we can use the method of *cylindrical shells*; rather than finding the volume by stacking discs/washers of variable radius we stack hollow cylinders of variable height inside each other.

Let S be a solid formed by rotating the area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ about the line $x = c$. The volume of S calculated using cylindrical shells is

$$V = \int_a^b 2\pi r h\,dx,$$

where $h = f(x) - g(x)$ is the height of a shell (assuming $f(x) \ge g(x)$ for $x \in [a, b]$) and r is the radius of a shell: $r = x - c$ if $c \le a < b$ and $r = c - x$ if $a < b \le c$. If the region is rotated about the line $y = c$ then exchange the roles of x and y above and integrate with respect to y. If S is the region between the curve

$y = f(x)$ and the x-axis on the interval $[a, b]$, with $f(x) \geq 0$ on $[a, b]$, then $g(x) = 0$, so $h = f(x)$; if the axis of rotation is the y-axis (the line $x = 0$) then $r = x$, so in this special case

$$V = \int_a^b 2\pi rh\,dx = \int_a^b 2\pi x f(x)\,dx.$$

Example. *Use cylindrical shells to find the volume of the solid generated by rotating the region bounded by $x = 2y^2$ and $x = y^2 + 1$, about the line $y = -2$.*

Solution. Since the axis of rotation is of the form $y = c$ the integral will be with respect to y. The curves $x = 2y^2$ and $x = y^2+1$ intersect when $2y^2 = y^2+1 \iff y^2-1 = 0 \iff y = \pm 1$, so the limits of integration are -1 and 1. On the interval $[-1, 1]$ we have $y^2+1 \geq 2y^2$, so the height of a shell is $h = y^2+1-2y^2 = 1-y^2$. To find r note that $c = -2 \leq a = -1 < b = 1$, so the definition above tells us $r = y - (-2) = y + 2$. Hence

$$\begin{aligned} V &= \int_a^b 2\pi rh\,dy = \int_{-1}^1 2\pi(y+2)(1-y^2)\ dy = 2\pi \int_{-1}^1 -y^3 - 2y^2 + y + 2\,dy \\ &= 2\pi\left[-\frac{y^4}{4} - \frac{2}{3}y^3 + \frac{y^2}{2} + 2y\right]_{-1}^1 = 2\pi\left(\left(-\frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2\right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2\right)\right) = \frac{16\pi}{3}. \end{aligned}$$

□

To decide whether cylindrical shells or the disc/washers method is better to compute a given volume one may have to simply try setting up each integral; in the tests and exams you will be told which method to use.

Example. *Consider the region R enclosed by the curves $y = x^3$ and $y = x^2$.*

(a) Use the disc/washer method to find the volume of the solid formed by rotating S about the line $y = -2$.
(b) Use the method of cylindrical shells to find the volume of the solid formed by rotating S about the y-axis.

Solution. (a) Since the axis of rotation is $y = -2$ we integrate with respect to x. The curves intersect when $x^2 = x^3 \iff x = 0$ or $x = 1$, which gives the limits of integration. Testing with $x = 1/2 \in [0, 1]$ we see that $x^2 \geq x^3$ on the interval $[0, 1]$, so outer radius is $x^2 - (-2)$ and inner radius $x^3 - (-2)$. Hence the area of a washer is $A(x) = \pi(x^2+2)^2 - \pi(x^3+2)^2 = \pi(x^4 + 4x^2 - x^6 - 4x^3)$. Therefore the volume is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^4 + 4x^2 - x^6 - 4x^3)\,dx = \pi\left[\frac{x^5}{5} + \frac{4}{3}x^3 - \frac{x^7}{7} - x^4\right]_0^1 = \pi\left(\frac{1}{5} + \frac{4}{3} - \frac{1}{7} - 1\right) = \frac{41}{105}.$$

(b) In this case we also integrate with respect to x. The radius of a shell is $r = x - 0$ and the height of a shell is $x^2 - x^3$ (since we know from (a) that x^2 is above x^3 on the interval $[0, 1]$). Therefore the volume is

$$V = \int_0^1 2\pi rh\,dx = 2\pi\int_0^1 x(x^2 - x^3)\,dx = 2\pi\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{\pi}{10}.$$

□

Work. For a constant force F to move an object the distance from $x = a$ to $x = b$ the *work done* is equal to the force multiplied by the distance: $W = F(b - a)$. This can be interpreted as the area under the graph of distance on the x-axis and force on the y-axis. If the force $F = f(x)$ changes as the distance x changes the work done is still the area under the graph of x against $f(x)$, so we arrive at the following definition.

Definition. *If an object moves from $x = a$ to $x = b$ under the action of a force $f(x)$ (depending on x) the work done is $W = \int_a^b f(x)\,dx$.*

If a cable hangs vertically and is pulled in from the top the force needed to pull the cable decreases, since the force needed is equal to the weight of the part of the cable which is still hanging down. The work required to pull in a vertically hanging cable of length l and mass m is $W = \int_0^l \frac{mg}{l}x\,dx$, where g is the acceleration due to gravity ($\frac{mg}{l}$ is the weight of the cable per unit length).

Example. *A rope weighing 90 Newtons, of length 20 metres, is used to raise a bucket filled with water. The weight of the bucket is 10 Newtons and the weight of water in the bucket before it is raised is 20 Newtons.*

(a) *Calculate the work done if the amount of water in the bucket is constant.*
(b) *Calculate the work done if the bucket is raised at a constant speed and water leaks from the bucket at a constant rate, so that the bucket becomes empty as it reaches the top.*

Solution. (a) Since the weight of bucket and water is constant the force needed to raise the bucket full of water is also constant, so the work done in raising the bucket and water is $Fd = 30(20) = 600$ joules. The total work done is the sum of the work on the bucket of water and the work on the rope. Our formula says the work on the rope is

$$\int_0^l \frac{mg}{l} x\,dx = \frac{90}{20}\int_0^{20} x\,dx = \frac{9}{2}\left[\frac{x^2}{2}\right]_0^{20} = 900$$

(we are given the weight of the rope mg, not the mass of the rope m). The total work is the sum $W = 600 + 900 = 1500$ joules.

(b) Since the bucket is raised at a constant speed we can write the weight of water remaining in the bucket as a function of the distance x the bucket has to be raised. The amount of water leaking per metre raised is $20/20 = 1$ (numerator is total water in bucket, denominator total metres raised); when the bucket has x metres remaining to be raised the water left in the bucket weighs $\frac{20}{20}(x) = x$ Newtons. Therefore the total work required is

$$\int_0^{20} 10 + x + \frac{90}{20}x\,dx = \left[10x + \frac{x^2}{2} + \frac{9x^2}{2}\right]_0^{20} = 200 + 200 + 900 = 1300.$$

(The first term in the integrand is the weight of the bucket, the second term is the weight of water, the third is the rope.)

□

Hooke's law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x, *i.e.* $f(x) = kx$ where $k > 0$ is a number called the spring constant. The work done in stretching a spring from a units passed natural length to b units passed its natural length is $W = \int_a^b kx\,dx$, where k is the spring constant.

Example. *(a) If the force required to maintain a spring* 0.5 *metres past its natural length is* 12 *Newtons find the work required to stretch the spring from its natural length to* 0.2 *metres past its natural length.*
(b) If work required to extend a spring from 1 *foot past its natural length to* 2 *feet past its natural length is* 9 *ft-lbs hom much force is required to hold the spring* 0.5 *feet past its natural length?*

Solution. (a) By Hooke's Law $f(x) = kx$, so $12 = 0.5k$ and therefore $k = 24$. Now our formula for work states the required work is

$$W = \int_a^b kx\,dx = \int_0^{0.2} 24x\,dx = 24\left[\frac{x^2}{2}\right]_0^{0.2} = \frac{12}{5}.$$

(b) Our formula for work states that

$$W = \int_a^b kx\,dx \implies 9 = \int_1^2 kx\,dx = \left[\frac{kx^2}{2}\right]_1^2 = k\left(2 - \frac{1}{2}\right) = \frac{3k}{2}.$$

Therefore $k = 2(9)/3 = 6$. The required force is therefore $kx = 6(0.5) = 3$.

□

The final type of example involves emptying water from a tank. Steps:

(1) let x be the depth below the surface of the tank, so $x = 0$ is the top of the tank and $x = d$ is the bottom of the tank;
(2) the volume of a slice, with thickness Δx, at a depth x is $A(x)\Delta x$, where $A(x)$ is the cross-sectional area of the slice;
(3) the weight of this slice is therefore $\rho A(x)\Delta x$, where ρ is the density of water;
(4) the distance h which each slice must be raised is x plus the height of a spout, if there is one;
(5) the work done emptying the tank is the integral $\int_0^d \rho h A(x)\,dx$.

Example. *A tank is the shape of a hemisphere of radius r metres (flat side to the top) and is filled with water. Find the work required to empty the tank:*

(a) by pumping all the water to the top of the tank;

(b) by pumping all the water out of a spout which rises 2 metres above the top of the tank.

Give your answers in terms of the density of water ρ.

Solution. (a) A slice of thickness Δx at a depth of x metres below the top of the tank is a circle of radius s, where $s^2 = r^2 - x^2$, so the weight of this slice is $\rho\pi s^2\Delta x = \rho\pi(r^2 - x^2)\Delta x$. This slice must be raised a distance x, so the work done is

$$\int_0^r \rho x\pi(r^2 - x^2)\,dx = \pi\rho\int_0^r r^2x - x^3\,dx = \pi\rho\left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r = \pi\rho\left(\frac{r^4}{2} - \frac{r^4}{4}\right) = \frac{\pi\rho r^4}{4}.$$

(b) The weight of a slice at depth x is the same as in (a), this time the $h = x + 2$, so the work done is

$$\int_0^r \rho(x+2)\pi(r^2 - x^2)\,dx = \pi\rho\int_0^r r^2x - x^3 + 2r^2 - 2x^2\,dx = \pi\rho\left[\frac{r^2x^2}{2} - \frac{x^4}{4} + 2r^2x - \frac{2x^3}{3}\right]_0^r$$
$$= \pi\rho\left(\frac{r^4}{2} - \frac{r^4}{4} + 2r^3 - \frac{2r^3}{3}\right) = \pi\rho\left(\frac{r^4}{4} + \frac{4r^3}{3}\right).$$

□

It is important to remember the difference between mass and weight in the questions on work.

Average value of a function. Our notion of average value of a function generalises the notion of the mean of n numbers $y_1, \ldots, y_n$: if $[a, b]$ can be divided in n subintervals of equal width, and $f(x_i^*) = y_i$ for any x_i^* in the ith subinterval, then the average value of f on $[a, b]$ is the mean of the numbers $y_1, \ldots, y_n$.

Definition. *The* average value *of the function f on the interval $[a, b]$ is*

$$\text{ave}_{[a,b]}(f) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The mean value theorem for integrals allows us to find an input c such that $f(c) = \text{ave}_{[a,b]}(f)$.

Theorem. *If f is continuous on $[a, b]$ then there exists $c \in [a, b]$ such that*

$$f(c) = \text{ave}_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. *(a) Find the average value of $f(x) = \dfrac{x^2}{(x^3+3)^2}$ on $[-1, 1]$.*

(b) Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Solution. (a) By definition

$$\text{ave}_{[-1,1]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-(-1)}\int_{-1}^1 \frac{x^2}{(x^3+3)^2}\,dx,$$

we evaluate this integral by substituting $u = x^3 + 3$, so $\frac{1}{3}du = x^2\,dx$ and $x = -1 \implies u = 2$, $x = 1 \implies u = 4$. So

$$\text{ave}_{[-1,1]}(f) = \frac{1}{2}\int_{-1}^1 \frac{x^2}{(x^3+3)^2}\,dx = \frac{1}{2}\int_2^4 \frac{1}{3}\frac{1}{u^2}\,du = \frac{1}{6}\left[-\frac{1}{u}\right]_2^4 = -\frac{1}{6}\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{24}.$$

(b) By definition

$$\text{ave}_{[0,b]}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b}\int_0^b 2 + 6x - 3x^2\,dx = \frac{1}{b}\left[2x + 3x^2 - x^3\right]_0^b = 2 + 3b - b^2,$$

so we must solve $3 = 2 + 3b - b^2$. By the quadratic formula $b = \frac{3\pm\sqrt{5}}{2}$; both values of b are valid since both are positive.

□

Part 3: Integration techniques

Integration by parts. The integration by parts formula is the result of integrating the product rule for derivatives. The formula is

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \quad \text{or} \quad \int u\,dv = uv - \int v\,du.$$

To integrate by parts one chooses a factor $u = f(x)$ in the integrand and computes $du = f'(x)\,dx$, the other factor in the integrand is called $dv = g'(x)\,dx$ and the antiderivative is $v = g(x)$ (any antiderivative works). The idea is that a difficult integral $\int u\,dv$ can be found using a simpler integral $\int v\,du$, so we aim to choose u and dv so that du is simpler than u and dv so that v is not more complicated than dv.

A useful rule of thumb is LIATE: choose u to be the type of function which appears as early on this list as possible: Logarithm, Inverse trigonometric, Algebraic (polynomial), Trigonometric, Exponential.

One may need to use other techniques, such as substitution, in combination with integration by parts.

Example. *Evaluate the integrals.*

(a) $\int x\sec^2(x)\,dx$

(b) $\int \sqrt{x}\ln(x)\,dx$

(c) $\int t^2\sin(t)\,dt$

(d) $\int \tan^{-1}(2y)\,dy$

Solution. (a) Let $u = x$ and $dv = \sec^2(x)\,dx$, so $du = dx$ and $v = \tan(x)$, so

$$\int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx = x\tan(x) - \ln|\sec(x)| + c$$

(the integral of $\tan(x)$ is one of the ones that should be remembered, but it can be found by writing $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and substituting $u = \cos(x)$).

(b) Let $u = \ln(x)$ and $dv = \sqrt{x}\,dx$, so $du = \frac{1}{x}\,dx$ and $v = \frac{2}{3}x^{3/2}$. Then

$$\int \sqrt{x}\ln(x)\,dx = \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + c.$$

(c) Let $u = t^2$ and $dv = \sin(t)\,dt$, so $du = 2t\,dt$ and $v = -\cos(t)\,dt$. Therefore

$$\int t^2\sin(t)\,dt = -t^2\cos(t) + \frac{1}{2}\int t\cos(t)\,dt.$$

To calculate the second integral use integration by parts again with $u_1 = t$ and $dv_1 = \cos(t)\,dt$, so $du_1 = dt$ and $v_1 = \sin(t)$. Therefore

$$\int t^2\sin(t)\,dt = -t^2\cos(t) + \frac{1}{2}\left(t\sin(t) - \int \sin(t)\,dt\right) = -t^2\cos(t) + \frac{1}{2}t\sin(t) + \frac{1}{2}\cos(t) + c.$$

(d) Let $u = \tan^{-1}(2y)$ and $dv = dy$, so $du = \frac{2}{1+4y^2}\,dy$ and $v = y$. Therefore

$$\int \tan^{-1}(2y)\,dy = y\tan^{-1}(2y) - \int \frac{2y}{1+4y^2}\,dy.$$

The second integral can be found by substituting $s = 1 + 4y^2$, so $ds = 8y\,dy$; hence

$$\int \tan^{-1}(2y)\,dy = y\tan^{-1}(2y) - \frac{1}{4}\int \frac{1}{s}\,ds = y\tan^{-1}(2y) - \frac{1}{4}\ln|s| + c = y\tan^{-1}(2y) - \frac{1}{4}\ln(1+4y^2) + c.$$

□

The formula for calculating a definite integral by parts is

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx.$$

Example. *Use the method of cylindrical shells to calculate the volume of the solid formed by rotating the region between the curves $y = e^{-x}$, $y = 0$, $x = -1$ and $x = 0$ about the line $x = 1$.*

Solution. The radius of a shell is $1 - x$ and the height of a shell is e^{-x}. Therefore the volume is

$$V = \int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx = 2\pi\left[-e^{-x}(1-x)\right]_{-1}^{0} - 2\pi\int_{-1}^{0} e^{-x}\,dx = 2\pi\left[-e^{-x}(1-x)\right]_{-1}^{0} - 2\pi\left[-e^{-x}\right]_{-1}^{0}$$
$$= 2\pi\left[xe^{-x}\right]_{-1}^{0} = 2\pi e.$$

We used integration by parts with $u = 1 - x$ and $dv = e^{-x}\,dx$, so $du = -dx$ and $v = -e^{-x}$. □

Integration by parts may be helpful to calculate the integrals of certain familiar functions, by taking $du = dx$. For example, to find $\int \ln(x)\,dx$ let $u = \ln(x)$ and $dv = dx$, so $du = \frac{1}{x}\,dx$ and $v = x$. Then

$$\int \ln(x)\,dx = x\ln(x) - \int \frac{x}{x}\,dx = x\ln(x) - x + c.$$

Occasionally integration by parts is useful when neither factor of the integrand becomes simpler when differentiating, for example $\int e^x \sin(x)\,dx$; in these cases applying integration by parts (maybe several times) results in an equation which can be solved for the required integral.

Trigonometric integrals. It is possible to calculate the integrals of many combinations of trigonometric functions using integration by substitution or integration by parts together with clever use of trigonometric identities.

The most important of the trigonometric identities are:

(1)
$$\sin^2(x)+\cos^2(x) = 1, \quad \sin(x+y) = \sin(x)\cos(y)+\cos(x)\sin(y), \quad \cos(x+y) = \cos(x)\cos(y)-\sin(x)\sin(y),$$

since all the identities used in this section can be deduced from these three. Dividing the first identity by $\sin^2(x)$ and $\cos^2(x)$ respectively gives

$$1 + \cot^2(x) = \csc^2(x), \quad \text{and} \quad \tan^2(x) + 1 = \sec^2(x).$$

Replacing y by $-y$ in the second and third identities of (1), and using that sin is an odd function and cos is an even function, gives

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) \quad \text{and} \quad \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$

Adding and subtracting the pairs of identities above which both contain a $\sin(x)\cos(y)$, $\cos(x)\cos(y)$, or $\sin(x)\sin(y)$ term we obtain

$$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right), \quad \cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right),$$

and

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right).$$

Taking $y = x$ in the second and third identities of (1) gives the double-angle identities

$$\sin(2x) = 2\sin(x)\cos(x), \quad \text{and } \cos(2x) = \cos^2(x) - \sin^2(x).$$

The expression for $\cos(2x)$ can be rearranged using $\sin^2(x) + \cos^2(x) = 1$ to get the half-angle identities

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right), \quad \text{and } \sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right).$$

It is probably easier to derive these identities from the basic ones given in (1) than to learn all of them. They will become familiar through frequent use.

The key thing to realise is that we can evaluate integrals such as $\int \sin(x)\,dx$ and $\int \cos(x)\,dx$ (and therefore $\int \sin(kx)\,dx$ and $\int \cos(kx)\,dx$), but we cannot evaluate $\int \sin^n(x)\,dx$ or $\int \cos^m(x)\,dx$ directly — it is almost impossible to find an antiderivative for the integrand by trial and error, and substituting $u = \sin(x)$ or $u = \cos(x)$ does not work (except in rare cases) because a new trigonometric function is introduced by du. These substitutions do work on the integrals $\int \sin^n(x)\cos(x)\,dx$ or $\int \cos^m(x)\sin(x)\,dx$. There are no substitutions that work on integrals of the form $\int \sin^{2n}(x)\cos^{2m}(x)\,dx$ — even if we use the identity $\sin^2(x) + \cos^2(x) = 1$ we will not be left with an odd power of sin or cos to use in du. For integrals involving sin and cos we aim to use trigonometric identities to write them in terms of the two above forms which can be integrated.

Example. *Evaluate the integrals.*

(a) $\displaystyle\int_0^{\pi/2} \sin^5(2x)\cos^4(2x)\,dx$

(b) $\displaystyle\int_0^{\pi} \sin^2(t)\cos^4(t)\,dt$

(c) $\displaystyle\int x\sin^3(x)\,dx$

(d) $\displaystyle\int \tan^2(x)\cos^3(x)\,dx$

(e) $\displaystyle\int \sin^2(3t)\cos^2(2t)\,dt$

Solution. (a) We leave only one factor of $\sin(2x)$ and convert the rest to cos, then substitute $u = \cos(2x)$, so $du = -\frac{1}{2}\sin(2x)\,dx$, and $x = 0 \implies u = 1$, $x = \pi/2 \implies u = -1$. So

$$\int_0^{\pi/2} \sin^5(2x)\cos^4(2x)\,dx = \int_0^{\pi/2} (1-\cos^2(2x))^2\cos^4(2x)\sin(2x)\,dx = \int_1^{-1} (1-u^2)u^4(-2du)$$
$$= 2\int_{-1}^{1} u^4 - u^6\,du = 2\left[\frac{u^5}{5} - \frac{u^7}{7}\right]_{-1}^{1} = 2\left(\left(\frac{1}{5}-\frac{1}{7}\right) - \left(-\frac{1}{5}+\frac{1}{7}\right)\right) = \frac{8}{35}.$$

(b) We use the identity for $\sin(t)\cos(t)$, then the half-angle identities (alternatively one could begin with the half-angle identities, but this seems to be more difficult). We get

$$\int_0^{\pi} \sin^2(t)\cos^4(t)\,dt = \int_0^{\pi} \frac{1}{4}\sin^2(2t)\cos^2(t)\,dt = \frac{1}{4}\int_0^{\pi} \frac{1}{2}(1-\cos(4t))\frac{1}{2}(1+\cos(2t))\,dt$$
$$= \frac{1}{16}\int_0^{\pi} 1 + \cos(2t) - \cos(4t) - \cos(4t)\cos(2t)\,dt$$
$$= \frac{1}{16}\int_0^{\pi} 1 + \cos(2t) - \cos(4t) - \frac{1}{2}\left(\cos(6t) + \cos(2t)\right)\,dt$$
$$= \frac{1}{16}\int 1 + \frac{1}{2}\cos(2t) - \cos(4t) - \frac{1}{2}\cos(6t)\,dt$$
$$= \frac{1}{16}\left[t + \frac{1}{4}\sin(2t) - \frac{1}{4}\sin(4t) - \frac{1}{12}\sin(6t)\right]_0^{\pi}$$
$$= \frac{1}{16}\left((\pi+0-0-0) - (0+0-0-0)\right) = \frac{\pi}{16}.$$

(c) This looks like an integration by parts question, so choose $u = x$ and $dv = \sin^3(x)\,dx$. Then $du = dx$; to calculate v:

$$\int \sin^3(x)\,dx = \int (1-\cos^2(x))\sin(x)\,dx = \int (1-t^2)(-dt) = \frac{t^3}{3} - t + c = \frac{\cos^3(x)}{3} - \cos(x) + c_1,$$

using the substitution $t = \cos(x)$ (u is already in use). Now let us calculate the integral in question:

$$\int x\sin^3(x)\,dx = x\left(\frac{\cos^3(x)}{3} - \cos(x)\right) - \int \frac{\cos^3(x)}{3} - \cos(x)\,dx.$$

(Remember that v can be any antiderivative of dv, so I have taken $c_1 = 0$.) We need to use another trigonometric identity to calculate the integral of the term involving $\cos^3(x)$, similar to how we found v

from dv write $\cos^3(x) = (1-\sin^2(x))\cos(x)$ and substitute $s = \sin(x)$, so $ds = \cos(x)\,dx$. Then

$$\begin{aligned}\int x\sin^3(x)\,dx &= x\left(\frac{\cos^3(x)}{3} - \cos(x)\right) - \int \frac{\cos^3(x)}{3} - \cos(x)\,dx \\ &= x\left(\frac{\cos^3(x)}{3} - \cos(x)\right) - \frac{1}{3}\int 1 - s^2\,ds + \sin(x) \\ &= x\left(\frac{\cos^3(x)}{3} - \cos(x)\right) - \frac{1}{3}\left(s - \frac{s^3}{3}\right) + c + \sin(x) \\ &= \frac{1}{3}x\cos^3(x) - x\cos(x) - \frac{1}{3}\sin(x) + \frac{1}{9}\sin^3(x) + \sin(x) + c \\ &= \frac{1}{3}x\cos^3(x) - x\cos(x) + \frac{2}{3}\sin(x) + \frac{1}{9}\sin^3(x) + c.\end{aligned}$$

(d) We have

$$\int \tan^2(x)\cos^3(x)\,dx = \int \frac{\sin^2(x)}{\cos^2(x)}\cos^3(x)\,dx = \int \sin^2(x)\cos(x)\,dx = \int u^2\,du = \frac{u^3}{3} + c = \frac{\sin^3(x)}{3} + c,$$

using the substitution $u = \sin(x)$, so $du = \cos(x)\,dx$.

(e) Since the variables of sin and cos are not equal we cannot use the identity involving $\sin(x)\cos(x)$; instead we first use the identity involving $\sin(x)\cos(y)$, then several other identities to make sure each term is easy to integrate:

$$\begin{aligned}\int \sin^2(3t)\cos^2(2t)\,dt &= \int \frac{1}{4}\left(\sin(5t) + \sin(t)\right)^2\,dt = \frac{1}{4}\int \sin^2(5t) + 2\sin(5t)\sin(t) + \sin^2(t)\,dt \\ &= \frac{1}{4}\int \frac{1}{2}(1-\cos(10t)) + (\cos(t) - \cos(6t)) + \frac{1}{2}(1-\cos(2t))\,dt \\ &= \frac{1}{4}\int 1 - \frac{1}{2}\cos(10t) + \cos(t) - \cos(6t) - \frac{1}{2}\cos(2t)\,dt \\ &= \frac{1}{4}\left(t - \frac{1}{20}\sin(10t) + \sin(t) - \frac{1}{6}\sin(6t) - \frac{1}{4}\sin(2t)\right) + c.\end{aligned}$$

□

To calculate integrals such as $\int \tan^m(x)\sec^n(x)\,dx$ we can often use a similar idea as for $\int \sin^m(x)\cos^n(x)\,dx$ when m or n is odd: save a factor of $\sec^2(x)$ and use $\tan^2(x) + 1 = \sec^2(x)$ to express all other factors in terms of $\tan(x)$, then substitute $u = \tan(x)$ so $du = \sec^2(x)\,dx$; alternatively, save a factor of $\sec(x)\tan(x)$ and use $\tan^2(x) + 1 = \sec^2(x)$ to express all other factors in terms of $\sec(x)$, then substitute $u = \sec(x)$ so $du = \sec(x)\tan(x)\,dx$.

A similar idea also works for certain integrals of the form $\int \csc^m(x)\cot^n(x)\,dx$: save a factor $\csc^2(x)$ and use $1 + \cot^2(x) = \csc^2(x)$ to express all other factors in terms of $\cot(x)$, then substitute $u = \cot(x)$, so $du = \csc^2(x)\,dx$; alternatively, save a factor $\csc(x)\cot(x)$ and use $1 + \cot^2(x) = \csc^2(x)$ to express all other factors in terms of $\csc(x)$, then substitute $u = \csc(x)$, so $-du = \csc(x)\cot(x)\,dx$.

Example. *(a)* $\int \sec^4(x)\tan^3(x)\,dx$

(b) $\int \tan^5(\theta)\sec^3(\theta)\,d\theta$

(c) $\int \cot^5(x)\csc^3(x)\,dx$

(d) $\int \cot^3(x)\,dx$

Solution. (a) Let $u = \tan(x)$, so $du = \sec^2(x)\,dx$, so

$$\begin{aligned}\int \sec^4(x)\tan^3(x)\,dx &= \int \sec^2(x)\tan^3(x)\sec^2(x)\,dx = \int (\tan^2(x)+1)\tan^3(x)\sec^2(x)\,dx = \int u^5 + u^3\,du \\ &= \frac{u^6}{6} + \frac{u^4}{4} + c = \frac{1}{6}\tan^6(x) + \frac{1}{4}\tan^4(x) + c.\end{aligned}$$

(b) This time, since the powers are odd, take $u = \sec(\theta)$ so $du = \sec(\theta)\tan(\theta)\,d\theta$. The remaining power of $\tan(\theta)$ is even, so we can use the identity $\tan^2(\theta) + 1 = \sec^2(\theta)$:

$$\begin{aligned}\int \tan^5(\theta)\sec^3(\theta)\,d\theta &= \int \tan^4(\theta)\sec^3(\theta)\sec(\theta)\tan(\theta)\,d\theta = \int (\sec^2(\theta)-1)^2\sec^2(\theta)\sec(\theta)\tan(\theta)\,d\theta \\ &= \int (u^2-1)^2u^2\,du = \int u^6 - 2u^4 + u^2\,du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c \\ &= \frac{1}{7}\sec^7(\theta) - \frac{2}{5}\sec^5(\theta) + \frac{1}{3}\sec^3(\theta) + c.\end{aligned}$$

(c) We save a factor of $\csc(x)\cot(x)$ and write the rest of the integrand in terms of $\csc(x)$ using the identity $1 + \cot^2(x) = \csc^2(x)$, then substitute $u = \csc(x)$, so $du = -\csc(x)\cot(x)\,dx$:

$$\begin{aligned}\int \cot^5(x)\csc^3(x)\,dx &= \int \cot^4(x)\csc^2(x)\csc(x)\cot(x)\,dx = \int (\csc^2(x)-1)^2\csc^2(x) csc(x)\cot(x)\,dx \\ &= -\int (u^2-1)^2u^2\,du = -\int u^6 - 2u^4 + u^2\,du = -\frac{u^7}{7} + \frac{2u^5}{5} - \frac{u^3}{3} + c \\ &= -\frac{1}{7}\csc^7(x) + \frac{2}{5}\csc^5(x) - \frac{1}{3}\csc^3(x) + c.\end{aligned}$$

(d) The identity $1+\cot^2(x) = \csc^2(x)$ can be used to rewrite this integral in a way which allows substitution:

$$\int \cot^3(x)\,dx = \int \cot(x)(\csc^2(x) - 1)\,dx = \int \cot(x)\csc^2(x) - \cot(x)\,dx.$$

The integral of $\cot(x)$ is known (it can easily be found by writing $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and substituting $u = \sin(x)$); since the derivative of $\cot(x)$ is $-\csc^2(x)$ the substitution $v = \cot(x)$, so $dv = -\csc^2(x)\,dx$, gives

$$\begin{aligned}\int \cot^3(x)\,dx &= \int \cot(x)\csc^2(x) - \cot(x)\,dx = \int -v\,dv + \int \cot(x)\,dx = -\frac{1}{2}v^2 - \ln|\sin(x)| + c \\ &= -\frac{1}{2}\cot^2(x) - \ln|\sin(x)| + c.\end{aligned}$$

□

Occasionally none of the above ideas will work — no trigonometric identities can be found to put the integrand in a form for which a substitution will work. In this case we must resort to integration by parts, algebraic tricks, or some other use of trigonometric identities.

Example. *(a)* $\int \sec(x)\,dx$

(b) $\int \csc^3(x)\,dx$

(c) $\int \tan^2(x)\sec(x)\,dx$

Solution. (a) Notice that $1 = \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$, so

$$\begin{aligned}\int \sec(x)\,dx &= \int \sec(x)\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}\,dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx \\ &= \int \frac{1}{u}\,du = \ln|u| + c = \ln|\sec(x)+\tan(x)| + c,\end{aligned}$$

using the substitution $u = \sec(x) + \tan(x)$, so $du = \sec^2(x) + \sec(x)\tan(x)\,dx$. (Note: a very similar trick works to calculate $\int \csc(x)\,dx$; check you understood this example by calculating $\int \csc(x)\,dx$.)

(b) A bit of trial and error should convince you that no trigonometric identity will help here. We use integration by parts with $u = \csc(x)$ and $dv = \csc^2(x)\,dx$, so $du = -\csc(x)\cot(x)\,dx$ and $v = -\cot(x)$.

Then

$$\begin{aligned}\int \csc^3(x)\,dx &= -\csc(x)\cot(x) - \int \csc(x)\cot^2(x)\,dx = -\csc(x)\cot(x) - \int \csc(x)(\csc^2(x)-1)\,dx\\ &= -\csc(x)\cot(x) + \int \csc(x)\,dx - \int \csc^3(x)\,dx\\ &= -\csc(x)\cot(x) + \ln|\csc(x)-\cot(x)| - \int \csc^3(x)\,dx\end{aligned}$$

(the technique to find the integral of $\csc(x)$ is the same as (a)). This identity can be solved for $\int \csc^3(x)\,dx$, giving

$$\int \csc^3(x)\,dx = -\frac{1}{2}\csc(x)\cot(x) + \frac{1}{2}\ln|\csc(x)-\cot(x)| + c.$$

(c) Using the identity $1+\tan^2(x) = \sec^2(x)$ we get

$$\int \tan^2(x)\sec(x)\,dx = \int \sec^3(x)\,dx - \int \sec(x)\,dx = \int \sec^3(x)\,dx - \ln|\sec(x)+\tan(x)| + c.$$

The integral $\int \sec^3(x)\,dx$ is very similar to the one in (b): use parts with $u = \sec(x)$ and $dv = \sec^2(x)\,dx$, so $du = \sec(x)\tan(x)\,dx$ and $v = \tan(x)$; therefore

$$\begin{aligned}\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx = \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)\,dx\\ &= \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx\\ &= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| - \int \sec^3(x)\,dx.\end{aligned}$$

Solving this identity for $\int \sec^3(x)\,dx$, gives

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

Therefore

$$\int \tan^2(x)\sec(x)\,dx = \int \sec^3(x)\,dx - \ln|\sec(x)+\tan(x)| + c = \frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

□

Integration by trigonometric substitution. Occasionally an integral $\int f(x)\,dx$ can be calculated by making a substitution $x = g(\theta)$ (this is different to the normal integration by substitution technique where we substitute $u = g(x)$); this kind of substitution is called an inverse substitution. If g is a trigonometric function then such a substitution allows us to take advantage of trigonometric identities to simplify the integrand.

The possible substitutions are:

$\sqrt{a^2-x^2}$	substitute $x = a\sin(\theta)$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	use the identity $1-\sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2+x^2}$	substitute $x = a\tan(\theta)$ with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	use the identity $1+\tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-a^2}$	substitute $x = a\sec(\theta)$ with $0 \le \theta < \frac{\pi}{2}$	use the identity $\sec^2(\theta)-1 = \tan^2(\theta)$.

The range of values for θ are chosen so that the trigonometric function is invertible, and ensure that the absolute value can be removed when simplifying the square root.

The above substitutions transform integrals involving the square root expressions into integrals involving trigonometric functions and powers of θ. The outcome of the integration will be a function of θ, which should be converted back to a function of x for the final answer. Suppose we made the substitution $x = a\sin(\theta)$ (the cases of sec or tan can be handled similarly), then $\theta = \sin^{-1}(x/a)$. If the result of the integration is another trigonometric function, say $\cot(\theta)$, then by drawing a right-angled triangle and labelling one of the angles θ we can express $\cot(\theta)$ in terms of x: our substitution was $\sin(\theta) = \frac{x}{a}$, since we know $\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$ label the side opposite θ as x and the hypotenuse as a. The adjacent side of the triangle is then $\sqrt{a^2-x^2}$ (by Pythagoras's theorem), and the expression for $\cot(\theta) = \frac{\text{adj}}{\text{opp}}$ is $\frac{x}{\sqrt{a^2-x^2}}$.

Example. *(a)* $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

(c) $\displaystyle\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

(d) $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

Solution. (a) Let $x = \sin(\theta)$, so $dx = \cos(\theta)\,d\theta$ and $x = 0 \implies \theta = 0$, $x = \sqrt{2}/2 \implies \theta = \pi/4$. Also

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)| = \cos(\theta),$$

the last equality because $\sin(\theta) \geq 0$ for $0 \leq \theta \leq \pi/4$. Hence

$$\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta = \int_0^{\pi/4} \frac{1}{2}(1-\cos(2\theta))\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/4}$$
$$= \frac{1}{2}\left(\left(\frac{\pi}{4} - \frac{1}{2}\right) - 0\right) = \frac{\pi}{8} - \frac{1}{4}.$$

Note that the trigonometric substitution resulted in an integral which needed a trigonometric identity to compute.

(b) Let $x = \sec(\theta)$ for $0 \leq \theta \leq \pi/2$. Then $dx = \sec(\theta)\tan(\theta)\,d\theta$ and

$$\sqrt{x^2-1} = \sqrt{\sec^2(\theta)-1} = \sqrt{\tan^2(\theta)} = |\tan(\theta)| = \tan(\theta),$$

the last equality resulting from our choice of possible values for θ. Hence

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{\sec(\theta)\tan(\theta)}{\sec^3(\theta)\tan(\theta)}\,d\theta = \int \cos^2(\theta)\,d\theta = \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + c = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + c.$$

We need to write the answer in terms of x, so we used the identity in the last equality so that all the quantities can be found from a right-angled triangle with hypotenuse x, adjacent 1 (since $\frac{x}{1} = \sec(\theta) = \frac{\text{hyp}}{\text{adj}}$), so the opposite side is $\sqrt{x^2-1}$. Hence $\cos(\theta) = \frac{1}{x}$ and $\sin(\theta) = \frac{\sqrt{x^2-1}}{x}$, and

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta) + c = \frac{1}{2}\sec^{-1}(x) + \frac{1}{2}\frac{\sqrt{x^2-1}}{x^2} + c.$$

(c) Let $2x = \tan(\theta)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $x = \frac{1}{2}\tan(\theta)$, $dx = \frac{1}{2}\sec^2(\theta)\,d\theta$ and

$$\sqrt{4x^2+1} = \sqrt{\tan^2(\theta)+1} = |\sec(\theta)| = \sec(\theta).$$

Then

$$\int \frac{1}{x\sqrt{4x^2+1}}\,dx = \int \frac{\frac{1}{2}\sec^2(\theta)}{\frac{1}{2}\tan(\theta)\sec(\theta)}\,d\theta = \int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta = \int \csc(\theta)\,d\theta = \ln|\csc(\theta) - \cot(\theta)| + c.$$

Now since $\frac{2x}{1} = \tan(\theta) = \frac{\text{opp}}{\text{adj}}$ the hypotenuse of the corresponding triangle has length $\sqrt{4x^2+1}$, so $\csc(\theta) = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{4x^2+1}}{2x}$ and $\csc(\theta) = \frac{\text{adj}}{\text{opp}} = \frac{1}{2x}$. Hence

$$\int \frac{1}{x\sqrt{4x^2+1}}\,dx = \ln|\csc(\theta) - \cot(\theta)| + c = \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + c.$$

(d) The denominator of the integrand is $\sqrt{4-x^2}^3$, which suggest we substitute $x = 2\sin(\theta)$, $-\pi/2 \le \theta \le \pi/2$, so $dx = 2\cos(\theta)$ and $\sqrt{4-x^2}^3 = 8\cos^3(\theta)$, since $\sin(\theta)$ is positive for the given range of θ. Hence

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2(\theta)}{8\cos^3(\theta)} 2\cos(\theta)\,d\theta = \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta$$
$$= \tan(\theta) - \theta + c = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + c.$$

The expression for $\tan(\theta)$ arises from $\frac{x}{2} = \sin(\theta) = \frac{\text{opp}}{\text{hyp}}$, so the adjacent side is $\sqrt{4-x^2}$.

□

Be aware that some integrals, such as $\int x\sqrt{4+x^2}\,dx$, can be solved by trigonometric substitution, but are more easily solved by a normal substitution, in this case $u = 4 + x^2$ so $\frac{1}{2}du = x\,dx$.

Integration by partial fractions. A rational function is the quotient of two polynomial functions: $\frac{P(x)}{Q(x)}$. Some rational functions can be integrated by substitution (if $P(x)$ is a scalar multiple of $Q'(x)$), but most of the time we need to use algebra to put the integrand in a form which is easier to integrate.

The first thing to check is that the degree of $P(x)$ (the highest power of x which appears) is less than the degree of $Q(x)$; if this is not the case then one must use polynomial division to write $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where S is a polynomial (and therefore easy to integrate) and the degree of $R(x)$ is less than the degree of $Q(x)$. A useful trick is

$$\frac{x}{x+q} = \frac{(q-q)+x}{x+q} = \frac{(x+q)-q}{x+q} = \frac{x+q}{x+q} + \frac{-q}{x+q} = 1 - \frac{q}{x+q},$$

which is easier than polynomial division and often puts the rational function in the form we need.

Now we focus on integrating $\frac{P(x)}{Q(x)}$ and assume that the degree of $P(x)$ is less than the degree of $Q(x)$. There are four cases to consider:

(a) all factors of $Q(x)$ are linear and distinct;
(b) all factors of $Q(x)$ are linear, but some appear more than once;
(c) at least one factor of $Q(x)$ is irreducible and quadratic, but none of the quadratic factors are repeated;
(d) $Q(x)$ has at least one repeated irreducible quadratic factor.

A linear factor has the form $ax + b$, while an irreducible quadratic factor has the form $ax^2 + bx + c$ with $b^2 - 4ac < 0$.

Case (a): $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$, each bracket is different. Then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n},$$

and the integral of each term on the right side is a logarithmic function:

$$\int \frac{A_i}{a_ix+b_i}\,dx = \frac{A_i}{a_i}\ln|a_ix+b_i| + c,$$

using the substitution $u = a_ix + b_i$, so $\frac{1}{a_i}du = dx$.

Case (b): $Q(x) = (a_1x+b_1)^{k_1}(a_2x+b_2)^{k_2}\cdots(a_nx+b_n)^{k_n}$, where some of the exponents k_i may be $2, 3, \ldots$. Then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} X_i,$$

and

$$X_i = \frac{A_1}{a_ix+b_i} + \frac{A_2}{(a_ix+b_i)^2} + \cdots + \frac{A_{k_i}}{(a_ix+b_i)^{k_i}}.$$

For example, the partial fraction decomposition for $\frac{x^2+1}{x^3(2x+1)(x-1)^2}$ is

$$\frac{x^2+1}{x^3(x+1)(x-1)^2} = \underbrace{\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3}}_{X_1} + \underbrace{\frac{B_1}{2x+1}}_{X_2} + \underbrace{\frac{C_1}{x-1} + \frac{C_2}{(x-1)^2}}_{X_3}.$$

The integral of each term is either a logarithmic function (as in case (a)) or a power function

$$\int \frac{A_i}{(a_ix+b_i)^k}\,dx = \int A_i(a_ix+b_i)^{-k}\,dx = \frac{A_i}{a_i(-k+1)}(a_ix+b_i)^{-k+1}+c, \quad k\neq 1,$$

using the substitution $u = a_ix+b_i$, so $\frac{1}{a_i}du = dx$, and the rule for integrating power functions.

Case (c): $Q(x) = (a_1x+b_1)^{k_1}\cdots(a_nx+b_n)^{k_n}(c_1x^2+d_1x+e_1)\cdots(c_mx^2+d_mx+e_m)$, where some of the exponents k_i may be $2,3,\ldots$ and the quadratic factors are irreducible and distinct. Then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} X_i + \frac{B_1x+C_1}{c_1x^2+d_1x+e_1}+\cdots+\frac{B_mx+C_m}{c_mx^2+d_mx+e_m},$$

where the X_i are as in case (b). For example, the partial fraction decomposition for $\frac{x^2+2}{x^3(x-1)(x^2+1)(x^2+x+1)}$ is

$$\frac{x^2+2}{x^3(x-1)(x^2+1)(x^2+x+1)} = \underbrace{\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}}_{X_1}+\underbrace{\frac{B_1}{x-1}}_{X_2}+\frac{C_1x+D_1}{x^2+1}+\frac{C_2x+D_2}{x^2+x+1}.$$

The integral of a term involving a quadratic factor is found by completing the square in the denominator, rearranging and using substitutions:

$$\int \frac{Cx+D}{ax^2+bx+c}\,dx = \int \frac{Cx+D}{(x+r)^2+s^2}\,dx = \int \frac{Eu+F}{u^2+s^2}\,du = \int \frac{Eu}{u^2+s^2}\,du + \int \frac{F}{u^2+s^2}\,du.$$

The first integral is a normal substitution, while the second is found by substituting $u = s\tan(\theta)$, so $du = s\sec^2(\theta)\,d\theta$ and

$$\int \frac{F}{u^2+s^2}\,du = \int \frac{s\sec^2(\theta)}{s^2(\tan^2(\theta)+1)}\,d\theta = \int \frac{s\sec^2(\theta)}{s^2\sec^2(\theta)}\,d\theta = \int \frac{1}{s}\,d\theta = \frac{\theta}{s}+c = \frac{1}{s}\tan^{-1}\left(\frac{u}{s}\right)+c.$$

Case (d): $Q(x) = (a_1x+b_1)^{k_1}\cdots(a_nx+b_n)^{k_n}(c_1x^2+d_1x+e_1)^{l_1}\cdots(c_mx^2+d_mx+e_m)^{l_m}$, where some of the exponents k_i and l_j may be $2,3,\ldots$ and the quadratic factors are irreducible. Then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j,$$

where the X_i are as in case (b) and Y_j are given by

$$Y_j = \frac{A_1x+B_1}{a_jx^2+b_jx+c_j}+\frac{A_2x+B_2}{(a_jx^2+b_jx+c_j)^2}+\cdots+\frac{A_{l_j}x+B_{l_j}}{(a_jx^2+b_jx+c_j)^{l_j}}.$$

For example, the partial fraction decomposition for $\frac{x^2+2}{x^3(x-1)(x^2+1)(x^2+x+1)^2}$ is

$$\frac{x^2+2}{x^3(x-1)(x^2+1)(x^2+x+1)^2} = \underbrace{\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}}_{X_1}+\underbrace{\frac{B_1}{x-1}}_{X_2}+\underbrace{\frac{C_1x+D_1}{x^2+1}}_{Y_1}+\underbrace{\frac{E_1x+F_1}{x^2+x+1}+\frac{E_2x+F_2}{(x^2+x+1)^2}}_{Y_2}.$$

The integral of each of the terms in the partial fraction decomposition can be found by a substitution or a trigonometric substitution.

Once the partial fraction decomposition has been determined one needs to find the value of the constants involved (the capital letters). The first step is to use algebra to remove the fractions, multiplying the equation given by the partial fraction decomposition by $Q(x)$. Then one deduces simultaneous equations involving the constants from this equation by plugging in values, or by multiplying everything out and equating the coefficients of each power of x.

Example. *(a)* $\displaystyle\int \frac{\sin^2(x)\cos(x)}{\sin^2(x)+\sin(x)}\,dx$

(b) $\displaystyle\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$

(c) $\displaystyle\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$

(d) $\displaystyle\int \frac{x^3+6x-2}{x^4+6x^2}\,dx$

Solution. (a) Substitute $u=\sin(x)$, so $du=\cos(x)\,dx$, then

$$\int \frac{\sin^2(x)\cos(x)}{\sin^2(x)+\sin(x)}\,dx = \int \frac{u^2}{u^2+u}\,du = \int \frac{u}{u+1}\,du = \int 1-\frac{1}{u+1}\,du = u-\ln|u+1|+c$$
$$= \sin(x)-\ln|\sin(x)+1|+c.$$

The third equality is the trick mentioned above, so that the rational function has the degree of the denominator larger than the degree of the numerator.

(b) The numerator and denominator have the same degree, so we use polynomial division first, which gives

$$\frac{x^3+4x^2+x-1}{x^3+x^2} = 1+\frac{3x^2+x-1}{x^3+x^2} = 1+\frac{3x^2+x-1}{x^2(x+1)}.$$

The partial fraction decomposition of the rational function is

$$\frac{3x^2+x-1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1} \iff 3x^2+x-1 = Ax(x+1)+B(x+1)+Cx^2;$$

substituting $x=0$ gives $B=-1$, $x=-1$ gives $C=1$, and $x=1$ then gives $A=2$. Hence

$$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx = \int_1^2 1+\frac{2}{x}-\frac{1}{x^2}+\frac{1}{x+1}\,dx = \left[x+2\ln|x|+\frac{1}{x}+\ln|x+1|\right]_1^2 = \frac{1}{2}+\ln(2)+\ln(3).$$

(c) Since $x^4+4x^2+3=(x^2+1)(x^2+3)$ the partial fraction decomposition is

$$\frac{x^3-2x^2+2x-5}{x^4+4x^2+3} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3} \iff x^3-2x^2+2x-5 = (Ax+B)(x^2+3)+(Cx+D)(x^2+1).$$

Solving the resulting system of equations gives $A=1/2,\ B=-3/2,\ C=1/2,\ D=-1/2$. Hence

$$\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx = \int \frac{\frac{1}{2}x-\frac{3}{2}}{x^2+1}+\frac{\frac{1}{2}x-\frac{1}{2}}{x^2+3}\,dx = \int \frac{\frac{1}{2}x}{x^2+1}-\frac{\frac{3}{2}}{x^2+1}+\frac{\frac{1}{2}x}{x^2+3}-\frac{\frac{1}{2}}{x^2+3}\,dx$$
$$= \frac{1}{4}\ln(x^2+1)-\frac{3}{2}\tan^{-1}(x)+\frac{1}{4}\ln(x^2+3)-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c.$$

(The first and third terms used the substitution $u=x^2+1$ and $u=x^2+3$ respectively, while the second and fourth terms use the formula given above which was found using a trigonometric substitution.)

(d) Since $x^4+6x^2=x^2(x^2+6)$ the partial fraction decomposition of the integrand is

$$\frac{x^3+6x-2}{x^4+6x^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+6} \iff x^3+6x-2 = (A+C)x^3+(B+D)x^2+6Ax+6B$$

(I have omitted some algebra). The coefficients for each power of x give four simultaneous equations, which solve to give $A=1,\ B=-1/3,\ C=0,\ D=1/3$. Hence

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,dx = \int \frac{1}{x}-\frac{\frac{1}{3}}{x^2}+\frac{\frac{1}{3}}{x^2+6}\,dx = \ln|x|+\frac{1}{3x}+\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right)+c.$$

□

Some integrals, such as (a) above, need a substitution before it is obvious to use partial fractions.

Summary of integration techniques. In an exam you are expected to choose an appropriate integration technique for each question. It is not possible to give a general algorithm which can be followed to determine an integration strategy, but the following guidelines may be helpful. To be able to follow these guidelines it is important to know the integrals of familiar functions (the table at the beginning of Section 7.5, which is reproduced below) so that you can recognise when an integrand is in a form which is easy to integrate.

(1) Simplify the integrand: move constant factors outside of the integrand, cancel out common factors, try multiplying any products (it is easier to find $\int \sqrt{x}+x^{3/2}\,dx$ than $\int \sqrt{x}(1+x)\,dx$), check if any trigonometric identities will help.

(2) Look for a substitution: if a composite function appears in the integrand then try making a substitution; often this will make it clearer what the next step is, for example a substitution might turn the integrand into a rational function which can be integrated using partial fractions. It is common for a question to require one or more substitutions along with other techniques.
(3) Look for something familiar: if the integrand looks like a rational function this suggests partial fractions; if the integrand contains a factor of $\sqrt{x^2+a^2}$ or something similar you should recognise this as requiring a trigonometric substitution (unless a normal substitution works); if the integrand is a product of trigonometric functions then consider using trigonometric identities, particularly if the functions involved are related to each others derivatives; if the integrand is a product of two functions, but not suitable for substitution, then try integration by parts; remember LIATE.
(4) Try something else: if none of the above have given an answer then keep trying; perhaps one needs to combine several of the above techniques, try using different trigonometric identities, a different substitution, or integration by parts with different choices of u and dv (remember it sometimes helps to choose $dv = dx$).

Above all, get as much practice as possible.

To make your work understandable it is important to mark clearly which integration technique you are using.

Be aware that there are many continuous functions which cannot be integrated using the techniques explained here. In fact every continuous function has an indefinite integral (FTC1 gives an expression for the antiderivative), but for most continuous functions this integral cannot be expressed in terms of the functions we have been working with. Some examples of integrals which cannot be expressed in terms of elementary functions are

$$\int \sin(x^2)\,dx, \quad \int \frac{1}{\ln(x)}\,dx, \quad \int \cos(e^x)\,dx, \quad \int \sqrt{x^3+1}\,dx.$$

Approximate integration. It may not be possible to evaluate a definite integral $\int_a^b f(x)\,dx$ using what we have learned; for example, f may be a function with no elementary antiderivative, so we cannot use FTC, or we may not have an expression for f which can be used, such as when f is a curve estimated from experimental data. When these situations arise it is useful to be able to approximate a definite integral.

The most basic approximations come directly from our definition of a definite integral: the expressions

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \quad \text{and} \quad R_n = \sum_{i=1}^{n} f(x_i)\Delta x, \quad \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x,$$

are the *left* and *right endpoint approximations* to $\int_a^b f(x)\,dx$.

The average of the left and right endpoint approximations is the *trapezoidal rule*:

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right),$$

where $\Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x$; this rule gets its name because the area under the graph on the interval $[x_{i-1}, x_i]$ is approximated by a trapezium with left side of length $f(x_{i-1})$, right side of length $f(x_i)$, and width Δx.

If one takes the height of a rectangle approximating the area under the graph on the interval $[x_{i-1}, x_i]$ to be the height of the graph $f(\overline{x_i})$ at the midpoint of the interval $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$ (instead of the left or right endpoints as in L_n and R_n) the approximation is called the *midpoint rule*:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left(f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + \cdots + f(\overline{x_{n-1}}) + f(\overline{x_n})\right),$$

where $\Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x$ and $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$.

The trapezoidal and midpoint rules use straight lines to approximate a curve $y = f(x)$, and therefore the area under the curve. A more sophisticated way of approximating a curve is using parabolas: given three points $y_{i-1} = f(x_{i-1}), y_i = f(x_i), y_{i+1} = f(x_{i+1})$ on a curve one can find numbers A, B, C such that the curve $y = Ax^2 + Bx + C$ passes through the points y_{i-1}, y_i, y_{i+1}, so the curve $y = Ax^2 + Bx + C$ should be a reasonable approximation of $y = f(x)$ on the interval $[x_{i-1}, x_{i+1}]$. Splitting the region $[a, b]$ in several intervals, and applying this idea to each interval, results in another approximation called *Simpson's rule*:

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right)$$

where n is even, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. Note the pattern in the sum: the odd-numbered terms are multiplied by 4, the first and last terms are multiplied by 1, all other even-numbered terms are multiplied by 2.

Example. *(a) Approximate* $\int_{-\pi}^{\pi} \sin^3(x)\,dx$ *using the midpoint rule with* $n = 4$.

(b) Approximate $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$ *using Simpson's rule and the trapezoid rule with* $n = 4$.

Solution. (a) Here $\Delta x = \pi/4$ and $x_i = -\pi + i\pi/4$. The midpoint rule gives

$$M_4 = \frac{\pi}{2}\left(\left(-\frac{1}{\sqrt{2}}\right)^3 + \left(-\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3\right) = 0.$$

This is an unusual example where the midpoint rule gives the exact answer, due to the symmetry of the integrand.

(b) Here $\Delta x = 1/4$ and $x_i = 0 + i/4$. Simpson's rule gives

$$S_4 = \frac{1}{3(4)}\left(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right) = \frac{1}{12}\left(0 + \frac{16}{17} + \frac{4}{5} + \frac{48}{25} + \frac{1}{2}\right) \approx 0.34676.$$

The trapezoid rule gives

$$T_4 = \frac{1}{2(4)}\left(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right) \approx 0.52015.$$

□

When using the above approximations it may be useful to know the size of the error in the approximation. The errors in each approximation to $\int_a^b f(x)\,dx$ are:

$$E_{T_n} = \int_a^b f(x)\,dx - T_n, \quad E_{M_n} = \int_a^b f(x)\,dx - M_n, \quad E_{S_n} = \int_a^b f(x)\,dx - S_n.$$

Careful study of each rule gives an expression for the largest possible magnitude of the error for a given n:

$$|E_{T_n}| \le \frac{K_T(b-a)^3}{12n^2}, \quad |E_{M_n}| \le \frac{K_M(b-a)^3}{24n^2}, \quad |E_{S_n}| \le \frac{K_S(b-a)^5}{180n^4},$$

where $|f''(x)| \le K_T$, $|f''(x)| \le K_M$, and $|f^{(4)}(x)| \le K_S$ for $a \le x \le b$ (K_T and K_M measure how far f is from being a straight line). Notice that the maximum error of M_n is half of the maximum error in T_n, so we expect that the midpoint rule is usually more accurate that the trapezoid rule, and that the maximum error in S_n is smaller than the other rules, particularly when n is large.

Example. *What is the largest possible error in an approximation to* $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$ *using the trapezoidal rule with* $n = 4$*? How large should* n *be to ensure the approximation using the midpoint rule is less than* $1/1000$*?*

Solution. Using the product rule we see that the second derivative of $\frac{x}{\sqrt{x^2+1}}$ is $\frac{2x(x^2-3)}{(x^2+1)^3}$; we want to find an upper bound for the absolute value of this expression when $0 \le x \le 1$. Using Calculus I techniques the numerator $2x(x^2-3)$ is decreasing on $[0, 1]$ and has a local minimum at $x = 1$; since the numerator is 0 when $x = 0$ we see that $|2x(x^2-3)| \le |2(1)(1^2-3)| = 4$. Since the denominator of the second derivative is always at least 1 we have shown that $|f''(x)| \le 4$ on $[0, 1]$. The largest possible error is

$$|E_{T_n}| \le \frac{K_T(b-a)^3}{12n^2} = \frac{4(1-0)^3}{12(4^2)} = \frac{1}{48} \approx 0.20833.$$

This approximation would not be very accurate because the error has the same order of magnitude as the approximate value.

Using the calculations above

$$\frac{1}{1000} < |E_{M_n}| \implies \frac{1}{1000} < \frac{4(1^3)}{24n^2} \implies n^2 > \frac{1000}{6} \approx 166.67,$$

so we should take $n > \sqrt{167} \approx 12.92$, so $n \geq 13$. □

L'Hospital's rule. The techniques from Calculus I may not work to calculate certain limits of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ (a can be a real number or $\pm\infty$) whenever $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or both limits are $\pm\infty$ (when this happens $\lim_{x\to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$).

L'Hospital's rule. *Suppose that f and g are differentiable functions and $g'(x) \neq 0$ on an open interval containing a (except possibly at a; $a = \pm\infty$ and one-sided limits are also permitted). If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, or is $\pm\infty$.

If the limit on the right side above is again an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then it can be calculated by applying l'Hospital's rule a second time.

In Calculus I we saw that close to $x = a$ the curves $f(x)$ and $g(x)$ are reasonably approximated by the lines through $f(a)$ and $g(a)$ with slope $f'(a)$ and $g'(a)$ respectively; l'Hospital's rule states that the ratio between these approximations is the same as the ratio between the original functions when we take the limit as x goes to a.

In order to use l'Hospital's rule effectively it is important to be able to recognise quickly determine the value of limits. Here are a few important ones (do not try to learn them, instead make sure you understand why these statements are true):

$$\lim_{x\to\infty} x^p = \infty \ (p > 0), \quad \lim_{x\to\infty} x^{-p} = 0 \ (p > 0), \quad \lim_{x\to\infty} \ln(x) = \infty, \quad \lim_{x\to\infty} a^x = \begin{cases} 0 & \text{if } 0 < a < 1 \\ \infty & \text{if } a > 1 \end{cases}.$$

The third limit will often be used with $a = e$.

Example. *Calculate the following limits.*

(a) $\lim_{x\to 0} \dfrac{\tan(3x)}{\sin(2x)}$

(b) $\lim_{x\to 0} \dfrac{x^2}{1-\cos(x)}$

(c) $\lim_{x\to\infty} \dfrac{x^n}{e^x}$ *(n a positive integer)*

(d) $\lim_{x\to\infty} x \sin\left(\dfrac{\pi}{x}\right)$

Solution. (a) This limit is type $\frac{0}{0}$. Applying l'Hospital's rule:

$$\lim_{x\to 0} \frac{\tan(3x)}{\sin(2x)} \overset{H}{=} \lim_{x\to 0} \frac{3\sec^2(3x)}{2\cos(2x)} = \frac{3(1)^2}{2(1)} = \frac{3}{2}.$$

(b) This limit is type $\frac{0}{0}$. Applying l'Hospital's rule:

$$\lim_{x\to 0} \frac{x^2}{1-\cos(x)} \overset{H}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \overset{H}{=} \lim_{x\to 0} \frac{2}{\cos(x)} = 2.$$

Note: we used l'Hospital's rule to calculate $\lim_{x\to 0} \frac{2x}{\sin(x)}$; alternatively one can remember that we showed $\lim_{x\to 0} \frac{\sin(x)}{x} = 0$ when calculating the derivatives of trigonometric functions.

(c) This limit is type $\frac{\infty}{\infty}$. Applying l'Hospital's rule gives

$$\lim_{x\to\infty} \frac{x^n}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{nx^{n-1}}{e^x},$$

which is again type $\frac{\infty}{\infty}$ unless $n-1=0$. If we apply l'Hospital's rule n times the power of x in the numerator will reduce to 0:

$$\lim_{x\to\infty} \frac{x^n}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \overset{H}{=} \cdots \overset{H}{=} \lim_{x\to\infty} \frac{n!x^1}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{n!x^0}{e^x} = 0.$$

This example shows that as $x \to \infty$ the function e^x becomes larger than x^n, even if n is very large. This rule of thumb can serve as a useful guide when computing limits, as well as in the next section.

(d) This does not appear to be in the correct form to apply l'Hospital's rule, but if we let $t = 1/x$ then $t \to 0 \iff x \to \infty$, and

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{t\to 0} \frac{\sin(\pi t)}{t} \overset{H}{=} \lim_{t\to 0} \frac{\pi\cos(\pi t)}{1} = \pi.$$

The discussion of indeterminate products below gives another approach to this question.

□

To make your work understandable it is important to mark clearly where you use l'Hospital's rule.

Certain other indeterminate limits can be put into a form to which l'Hospital's rule applies by using algebra.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)g(x)$ is called an *indeterminate product* of type $0\cdot\infty$ (notice that $0\cdot 0$ and $\pm\infty\cdot\pm\infty$ are not indeterminate products). This limit can be calculate by writing

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}},$$

which is of type $\frac{0}{0}$ and can be found by applying l'Hospital's rule.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x) - g(x)$ is called an *indeterminate difference* of type $\infty - \infty$ (notice that $0-0$, $\infty+\infty$, and several others, are not indeterminate). This limit can be calculated by forcing it to be an indeterminate product or quotient, for example by writing

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a} f(x)\left(1 - \frac{g(x)}{f(x)}\right)$$

or bringing fractions to a common denominator, and applying l'Hospital's rule.

The limit $\lim_{x\to a} (f(x))^{g(x)}$ gives rise to several indeterminate forms: types 0^0, ∞^0 and 1^∞ (other types such as b^∞ or b^0 with $b \neq 1$, or ∞^∞, are not indeterminate). These limits can be found by letting $y = (f(x))^{g(x)}$, so that $\lim_{x\to a}\ln(y) = \lim_{x\to a} g(x)\ln(f(x))$, calculating the limit of the indeterminate product as above, then finding $\lim_{x\to a} y$ using

$$\lim_{x\to a} y = \lim_{x\to a} e^{\ln(y)} = e^{\lim_{x\to a} g(x)\ln(f(x))}.$$

Example. *Calculate the following limits.*

(a) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$

(b) $\lim_{x\to\infty} (x - \ln(x))$

(c) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\tan^{-1}(x)}\right)$

(d) $\lim_{x\to\infty} x^{e^{-x}}$

Solution. (a) Type $\infty\cdot 0$. Rearrange to type $\frac{\infty}{\infty}$ and apply l'Hospital's rule:

$$\lim_{x\to\infty} \sqrt{x}e^{-x/2} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}e^{x/2}} = 0.$$

(b) Type $\infty - \infty$. Factoring gives $\lim_{x\to\infty}(x - \ln(x)) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right)$. By l'Hospital's rule

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

so $\lim_{x\to\infty}(x - \ln(x)) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right) = \infty$ as it is the product of a term with infinite limit and a term with nonzero finite limit.

(c) Type $\infty - \infty$. We bring to a common denominator and use l'Hospital's rule twice:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\tan^{-1}(x)}\right) = \lim_{x\to 0^+} \frac{\tan^{-1}(x) - x}{x\tan^{-1}(x)} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x^2+1} - 1}{\frac{x}{x^2+1} + \tan^{-1}(x)} = \lim_{x\to 0^+} \frac{-x^2}{x + (x^2+1)\tan^{-1}(x)}$$
$$\overset{H}{=} \lim_{x\to 0^+} \frac{-2x}{1 + 1 + 2x\tan^{-1}(x)} = \frac{0}{2+0} = 0.$$

(d) Let $y = x^{e^{-x}}$, so $\ln(y) = e^{-x}\ln(x)$ and by l'Hospital's rule

$$\lim_{x\to\infty} e^{-x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0.$$

Hence

$$\lim_{x\to\infty} x^{e^{-x}} = \lim_{x\to\infty} e^{\ln(y)} = e^0 = 1.$$

□

Be aware that if the conditions for l'Hospital's rule are not satisfied then attempting to apply the rule may result in an incorrect limit. Do not confuse l'Hospital's rule with the quotient rule for derivatives. Finally, consider the limit $\lim_{x\to 0} \frac{\sin(x)}{x}$; l'Hospital's rule tells us that this limit is equal to $\lim_{x\to 0} \frac{\cos(x)}{1} = 1$, using that the derivative of $\sin(x)$ is $\cos(x)$. In Calculus I we proved this limit in order to show that $\frac{d}{dx}\sin(x) = \cos(x)$, so we cannot use l'Hospital's rule to discover the value of this limit for the first time; however it can be used if you forget the value of this limit.

Improper integrals. An improper integral can be thought of as calculating the area under a function, where the region under the function extends infinitely far in either the vertical or horizontal direction. We will see that it is possible for the area of such a region to be finite or infinite.

Definition. *An improper integral of type 1 is an integral over an infinite interval; that is, one (or both) of the limits of integration is infinite.*

If $\int_a^t f(x)\,dx$ exists for all $t \geq a$ then we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided the limit exists (i.e. the limit is finite).

If $\int_t^b f(x)\,dx$ exists for all $t \leq b$ then we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

provided the limit exists (i.e. the limit is finite).

The above improper integrals are called convergent *if the limit involved exists and* divergent *if the limit does not exist. If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then we define*

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

It can be shown that any real number a can be used to split the above integral.

Example. *Decide if the improper integrals are divergent or convergent. If they are convergent find their value.*

(a) $\displaystyle\int_1^\infty \frac{\ln(x)}{x^4}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{4x^2+4x+5}\,dx$

(c) $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ *($p > 0$)*

Solution. (a) We use integration by parts with $u = \ln(x)$ and $dv = \frac{1}{x^4}\,dx$ to calculate the integral in the definition:

$$\int_1^\infty \frac{\ln(x)}{x^4}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x^4}\,dx = \lim_{t\to\infty}\left(\left[-\frac{\ln(x)}{3x^3}\right]_1^t + \int_1^t \frac{1}{3x^4}\,dx\right) = \lim_{t\to\infty}\left(\left[-\frac{\ln(x)}{3x^3}\right]_1^t + \left[-\frac{1}{9x^3}\right]_1^t\right)$$

$$= \lim_{t\to\infty}\left(-\frac{\ln(t)}{3t^3} + \left(-\frac{1}{9t^3}+\frac{1}{9}\right)\right) \overset{H}{=} \lim_{t\to\infty}\left(-\frac{1}{9t^3} + \left(-\frac{1}{9t^3}+\frac{1}{9}\right)\right) = \frac{1}{9}.$$

Note the use of l'Hospital's rule to calculate the limit of the first term.

(b) Let $u = 2x+1$, so $du = 2dx$, and

$$\int_{-\infty}^{\infty} \frac{1}{4x^2+4x+5}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{u^2+4}\,du = \frac{1}{2}\int_{-\infty}^{0}\frac{1}{u^2+4}\,du + \frac{1}{2}\int_{0}^{\infty}\frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\lim_{t\to-\infty}\int_t^0 \frac{1}{u^2+4}\,du + \frac{1}{2}\lim_{t\to\infty}\int_0^t \frac{1}{u^2+4}\,du$$

$$= \frac{1}{2}\lim_{t\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_t^0 + \frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_0^t = \frac{\pi}{4}.$$

(c) When $p = 1$ we have

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\ln|x|\right]_1^t = \lim_{t\to\infty}(\ln(t)-\ln(1)) = \infty,$$

so the improper integral is divergent when $p = 1$. If $p \neq 1$ then

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right).$$

If $p > 1$ then $\frac{1}{t^{p-1}} \to 0$ as $t \to \infty$, so the improper integral converges; if $p < 1$ then $\frac{1}{t^{p-1}} \to \infty$ as $t \to \infty$, so the improper integral diverges. In summary, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges when $0 < p \le 1$ and converges to $\frac{1}{p-1}$ when $p > 1$.

□

Improper integrals of type 1 correspond to the area of a region which extends infinitely in the horizontal direction. The second type correspond to the area of a region which extends infinitely in the vertical direction, which occurs when the integrand has an infinite discontinuity.

Definition. *An improper integral of type 2 is an integral involving an integrand with an infinite discontinuity.*

If f is continuous on $[a, b)$ and is discontinuous at b then we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx,$$

provided the limit exists (i.e. the limit is finite).

If f is continuous on $(a, b]$ and is discontinuous at a then we define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx,$$

provided the limit exists (i.e. the limit is finite).

The improper integral $\int_a^b f(x)\,dx$ is called convergent *if the limit involved exists and* divergent *if the limit does not exist. If f has a discontinuity at c, with $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note that in the above definition the limit always approaches the endpoint of the interval from within the interval.

Example. *Decide if the improper integrals are divergent or convergent. If they are convergent find their value.*

(a) $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$

(b) $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

Solution. (a) The integrand has an infinite discontinuity at $x = 2/3$. Using the definition

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t\to\frac{2}{3}^-} \int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to\frac{2}{3}^-} \left[-\frac{1}{3}\ln|2-3x|\right]_0^t = \lim_{t\to\frac{2}{3}^-} (\ln|2-3t| - \ln(2)) = \infty.$$

Since the left part of $\int_0^1 \frac{1}{2-3x}\,dx$ diverges the integral in question diverges.

(b) The integrand has an infinite discontinuity at 0, so

$$\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 x^{1/2} - x^{-1/2}\,dx = \lim_{t\to 0^+} \left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_t^1 = -\frac{4}{3}.$$

(c) Note that if $p \leq 0$ the integral is not improper. When $p = 1$ the improper integral diverges, since

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} [\ln|x|]_t^1 = \lim_{t\to 0^+} (\ln(1) - \ln(t)) = \infty.$$

When $p \neq 1$ we have, similarly to the calculation given above,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \frac{1}{1-p}\left(1 - \frac{1}{t^{p-1}}\right).$$

If $p > 1$ then $p - 1 > 0$, so $\frac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$ and the integral diverges. If $0 < p < 1$ then $p - 1 < 0$, so $\frac{1}{t^{p-1}} \to 0$ as $t \to 0^+$, so the integral converges to $\frac{1}{1-p}$. In summary, $\int_0^1 \frac{1}{x^p}\,dx$ diverges when $p \geq 1$ and converges to $\frac{1}{1-p}$ when $0 < p < 1$.

□

If you are asked to calculate a definite integral you are expected to decide if the integral is improper or not, and use the appropriate definition to check if any improper integral converges or diverges.

It may not be possible to calculate the exact value of an improper integral, but sometimes one can still decide if an improper integral converges or diverges by comparing it with another improper integral which is known to converge or diverge. Geometrically this corresponds to comparing an area which may or may not be infinite with another area which is known to be finite or infinite. The integrals $\int_a^b \frac{1}{x^p}\,dx$ are often useful for these comparisons.

Theorem. *Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

(a) If $\int_a^\infty f(x)\,dx$ is convergent then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.

A similar statement holds for improper integrals over the interval $(-\infty, b]$.

Now suppose f and g are continuous on $[a, b)$ and discontinuous at b, and $f(x) \geq g(x) \geq 0$ for $x \in [a, b)$.

(a) If $\int_a^b f(x)\,dx$ is convergent then $\int_a^b g(x)\,dx$ is convergent.

(b) If $\int_a^b g(x)\,dx$ is divergent then $\int_a^b f(x)\,dx$ is divergent.

A similar statement holds for improper integrals on $[a, b]$ with a discontinuity at a.

Example. *Use the comparison theorem to decide if the given integrals are convergent or divergent.*

(a) $\displaystyle\int_1^\infty \frac{2+\sin(x)}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^4}}\,dx$

Solution. (a) Note that $\frac{2+\sin(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}}$ for $x \in [1,\infty)$ (since $-1 \leq \sin(x)$). We know $\int_1^\infty \frac{1}{x^{1/2}}\,dx$ diverges, so the integral in question is divergent by the comparison theorem.

(b) Since $1+x^4 > x^4$ and $\sqrt{x}$ is an increasing function we have $\frac{1}{\sqrt{1+x^4}} < \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}$ for $x \in [1,\infty)$. The improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, so the integral in question is also convergent by the comparison theorem.

□

Part 4: Further applications of integration

Arc length. The length of a straight line segment is given by the distance formula from Calculus I. If a curve cannot be split into a number of straight lines then it is not immediately obvious how to define the length of the curve. The length of a curve is defined by approximating the curve by a straight line on each interval $[x_{i-1}, x_i]$, adding the lengths of these lines, and taking the limit as the number of intervals goes to infinity.

If f' is continuous on $[a,b]$ then the length of the curve $y = f(x)$ on the interval $[a,b]$ is

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

It may be useful to have a function which expresses the distance from a point $P(a, f(a))$ on the curve $y = f(x)$ to any other point on the curve. The *arc length function* for the distance along the curve $y = f(x)$ from the point $P(a, f(a))$ to another point $Q(x, f(x))$ is

$$s(x) = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Example. *(a) Find the length of the curve* $12x = 4y^3 + \dfrac{3}{y}$, $1 \leq y \leq 3$.

(b) Find the arc length function for the curve $y = 4(x-1)^{3/2}$ *with starting point* $x = 1$.

Solution. (a) Since $x = \frac{1}{3}y^3 + \frac{1}{4y}$ we have

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \implies 1+\left(\frac{dx}{dy}\right)^2 = 1 + y^4 - \frac{1}{2} + \frac{1}{16y^4} = \left(y^2 + \frac{1}{4y^2}\right)^2.$$

Hence the length of the curve is

$$L = \int_1^3 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\,dy = \int_1^3 y^2 + \frac{1}{4y^2}\,dy = \left[\frac{y^3}{3} - \frac{1}{4y}\right]_1^3 = \left(\left(9-\frac{1}{12}\right) - \left(\frac{1}{3}-\frac{1}{4}\right)\right) = \frac{53}{6}.$$

Note that the first equality in evaluating the integral is only valid because $y^2 + \frac{1}{4y^2}$ is positive on the interval $[1,3]$.

(b) We have

$$\frac{dy}{dx} = 6(x-1)^{1/2} \implies 1+\left(\frac{dy}{dx}\right)^2 = 1 + 36(x-1) = 36x - 35.$$

Therefore the arc length function with starting point $x = 1$ is

$$s(x) = \int_1^x \sqrt{36t-35}\,dt = \left[\frac{2}{3}(36)(36t-35)^{3/2}\right]_1^x = 24(36x-35)^{3/2} - 24.$$

□

Area of a surface of revolution. A surface of revolution is formed when a curve is rotated about an axis (it is the boundary of the solid of revolution we considered earlier). If the curve is a straight line of length h, parallel to the axis with distance r, then the surface of revolution is the surface area of a cylinder of radius r and height h, which we know to have surface area $2\pi rh$ (cut the surface along the straight line and unfold the surface into a rectangle with sides h and $2\pi r$). If the curve is a straight line not parallel to the axis then the area it generates is the surface area of (a frustum of) a cone. The surface area of a cone of base radius r and slant height l can be found by cutting the cone from the apex to the base, flattening the cone into the arc of a circle with radius l, arc length $2\pi r$, central angle $\theta = \frac{2\pi r}{l}$, therefore area πrl (given by the formula for sector area). The surface area of a frustum of a cone with slant height l and lower and upper radii r_1 and r_2 is $2\pi rl$, where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius; this formula can be found by writing the surface area of a frustum as the difference between surface areas of two cones.

To define the area generated by rotating any curve about an axis we break the curve in segments and approximate the area generated by each segment as the surface are of a cylinder or (a frustum of) a cone.

Definition. *If $f(x) \geq 0$ for $a \leq x \leq b$ and f' is continuous then the area of the surface obtained by rotating the curve $y = f(x)$ about the x-axis, between $x = a$ and $x = b$, is*

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If the curve is given as $x = g(y)$, $c \leq y \leq d$, then the surface formed by rotating the curve around the x-axis is

$$S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

For rotation about the y-axis exchange the roles of x and y above.

The first of the above formulas for surface area generated by rotation about the x-axis is sometimes given as $S = \int_a^b 2\pi y\, ds$, where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ is the differential coming from the derivative of the arc length function.

Example. *Find the area of the surface obtained by rotating the given curve about the axis.*
(a) $y = \cos\left(\dfrac{x}{2}\right)$, $0 \leq x \leq \pi$, *about the x-axis.*
(b) $y = \dfrac{1}{3}x^{3/2}$, $0 \leq x \leq 12$, *about the y-axis.*

Solution. (a) We have

$$\frac{dy}{dx} = -\frac{1}{2}\sin\left(\frac{x}{2}\right) \implies 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right).$$

To calculate the integral we first make the substitution $u = \sin\left(\frac{x}{2}\right)$, then calculate the resulting integral using the trigonometric substitution $u = 2\tan(\theta)$:

$$\begin{aligned} S &= \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi\int_0^\pi \cos\left(\frac{x}{2}\right)\sqrt{1 + \frac{1}{4}\sin^2\left(\frac{x}{2}\right)}\, dx = 2\pi\int_0^1 \sqrt{4 + u^2}\, du \\ &= 2\pi\int_{u=0}^{u=1} \sqrt{4 + 4\tan^2(\theta)}2\sec^2(\theta)\, d\theta = 8\pi\int_{u=0}^{u=1} \sec^3(\theta)\, d\theta \\ &= 4\pi\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right]_{u=0}^{u=1} = 4\pi\left[\frac{u}{2}\frac{\sqrt{u^2+4}}{2} + \ln\left(\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right)\right]_0^1 \\ &= 4\pi\left(\frac{\sqrt{5}}{4} + \ln\left(\frac{\sqrt{5}+1}{2}\right) - 0\right) = \pi\sqrt{5} + 4\pi\ln\left(\frac{\sqrt{5}+1}{2}\right). \end{aligned}$$

(The integral of $\sec^3(\theta)$ was computed above; I found it easier to write the trigonometric functions in terms of u before evaluating at the limits of integration.)

(b) Since we are rotating about the y-axis we exchange the roles of x and y in the definition; since the curve is given as $y = g(x)$ we use the second formula for the area. We have

$$\frac{dy}{dx} = \frac{1}{2}x^{1/2} \implies 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x}{4}.$$

It is convenient to use the substitution $u = x + 4$ to calculate the integral for surface area:

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^{12} x\sqrt{1 + \frac{x}{4}}\, dx = 2\pi \int_0^{12} x\frac{1}{2}\sqrt{4+x}\, dx \\ &= \pi \int_4^{16} (u-4)\sqrt{u}\, du = \pi \int_4^{16} u^{3/2} - 4u^{1/2}\, du = \pi \left[\frac{2u^{3/2}}{5} - \frac{8u^{3/2}}{3}\right]_4^{16} \\ &= \pi \left(\frac{2}{5}(992) - \frac{8}{3}(56)\right) = \frac{3712}{15}\pi. \end{aligned}$$

□

Part 5: Differential equations

Exponential growth and decay. If a quantity $y = f(x)$ grows (or decays) in proportion to its size this can be written as an equation: $\frac{dy}{dx} = kx$, where k is a number. This is our first example of a differential equation.

Since the above equation asks for a function whose derivative is a constant multiple of the function it is easy to see that exponential functions satisfy this differential equation. In fact there are no other solutions.

Theorem. *The only solutions of the differential equation* $\frac{dy}{dt} = ky$ *are exponential functions*

$$y(t) = Ce^{kt}, \quad \textit{where } C = y(0).$$

This theorem explains why quantities which grow or decay in proportion to their size are said to have exponential growth or decay. We have used the variable t because many natural occurrences of exponential growth and decay have time as the variable.

- Population growth: if $P(t)$ is the size of a population at time t then it often satisfies $\frac{dP}{dt} = kP$.
- Radioactive decay: let $m(t)$ be the mass of a radioactive substance remaining after time t, from an initial mass m_0. It has been found by experiments that $\frac{dm}{dt} = km$ for a constant $k < 0$ (depending on the substance); equivalently $-\frac{1}{m}\frac{dm}{dt}$ is constant. This means that a radioactive substance decays exponentially: $m(t) = m_0 e^{kt}$. Physicists express the rate of decay in terms of half-life $-\ln(2)/k$, a positive number which is the time taken for half the mass to decay.
- Newton's law of cooling: Newton discovered that the rate at which an object cools (or warms) is proportional to the difference between its temperature T and the surrounding temperature T_s; that is $\frac{dT}{dt} = k(T - T_s)$, where $T(t)$ is the temperature of an object at time t. Writing $y(t) = T(t) - T_s$ this relationship becomes $\frac{dy}{dt} = ky$, so the temperature of the object grows/decays exponentially.
- Continuously compounded interest: if an amount of money A_0 is invested at an interest rate r, and r is compounded n times in t years, then the value of the investment is given by $A(t) = A_0(1 + \frac{r}{n})^{nt}$. The interest paid increases as n grows larger (the more often interest is compounded), and as $n \to \infty$ (interest is compounded continuously) the value of the investment is given by

$$A(t) = \lim_{n\to\infty} A_0(1 + \frac{r}{n})^{nt} = A_0 \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right)^{rt} = A_0 e^{rt}, \quad (m = n/r).$$

  It follows that $\frac{dA}{dt} = rA(t)$; that is, with continuously compounded interest the rate of increase of an investment is proportional to the size of the investment.

Example. *(a) A curve passes through the point* $(0, 2)$ *and has the property that the slope of the curve at any point* P *is three times the* y*-coordinate of* P*. Find an equation for the curve.*

(b) How long will it take an investment to double in value if the interest rate is 6% *compounded continuously?*

Solution. (a) Suppose the curve is $y = f(x)$. We are told that $\frac{dy}{dx} = 3y = 3f(x)$, so (by the theorem) we must have $y = Ce^{kx}$. Since $C = Ce^0 = y(0) = 2$ we have $C = 2$, and differentiating gives $3y\frac{dy}{dx} = kCe^{kx} = 2ke^{kx}$, so $k = 3$. Hence the equation of the curve is $y = 2e^{3x}$.

(b) We know $A = A_0e^{rt}$, here $r = 0.06$, and we want to find t for which $A = 2A_0$:

$$2A_0 = A_0e^{0.06t} \implies e^{0.06t} = 2 \implies t = \frac{\ln(2)}{0.06} \approx 11.55,$$

so the investment will double in approximately 11.55 years.

□

Modelling with differential equations. A *differential equation* is an equation involving an unknown function and one or more of its derivatives; the order of a differential equation is the highest order derivative which occurs in the equation.

A function is a solution of a differential equation if the equation is satisfied when the function and its derivatives are substituted into the equation; we may need to find the general form of the solution to an equation or a particular solution which also satisfies a given initial condition (an initial condition gives a point which the curve given by the solution must pass through).

It can be very difficult to solve a differential equation, there may be no nice expression for a solution or the equation is too complicated to guess a solution. We have already solved differential equations of the form $f'(x) = kf(x)$ (exponential growth and decay) and many differential equations of the form $y' = f(x)$ (solving this equation amounts to calculating the integral of $f(x)$). Finding the solution to a differential equation often involves a bit of guesswork as well as a good understanding of the behaviour of the elementary functions and their derivatives.

Writing a differential equation which models a given situation requires a good understanding of how the situation is likely to behave at different times, and the natural constraints. For example, we used population size as an example of exponential growth, but that model does not account for the face that when most populations $P(t)$ have a critical level M above which they cannot support themselves; As t increases we want our model for $P(t)$ to reflect that $P(t)$ grows towards M, or decreases towards M. One possible model for population size is $P'(t) = kP(1 - \frac{P}{M})$, which is approximately the model we had above when P is small compared to M, when P is close to M $P(t)$ changes at a slow rate, and when $P(t)$ is larger than M it decreases towards M. This model reflects population size more accurately than the previous one.

Separable differential equations. A separable differential equation is a first-order differential equation which can be written in the form

$$\frac{dy}{dx} = g(x)f(y) = \frac{g(x)}{h(y)}, \quad h(y) = \frac{1}{f(y)},$$

where we assumed $f(y) \neq 0$ for the second equality. The solutions to such an equation can be found by writing in terms of differentials and integrating each side:

$$\frac{dy}{dx} = g(x)f(y) = \frac{g(x)}{h(y)} \implies h(y)\,dy = g(x)\,dx \implies \int h(y)\,dy = \int g(x)\,dx.$$

This procedure can be justified by taking the derivative of the latter equality with respect to x, using the chain rule.

Example. *Solve the differential equations.*

(a) $\dfrac{dy}{dx} = \dfrac{x\sin(x)}{y}$, $y(0) = -1$

(b) $xy' = y + xe^{y/x}$

Solution. (a) Separating the variables and integrating gives:

$$y\,dy = x\sin(x)\,dx \implies \int y\,dy = \int x\sin(x)\,dx \implies \frac{y^2}{2} = -x\cos(x) + \sin(x) + c,$$

using integration by parts. Using $y(0) = -1$ gives $\frac{1}{2}(-1)^2 = 0 + \sin(0) + c$, so $c = 1/2$. Hence $y^2 = -2x\cos(x) + 2\sin(x) + 1$, so y is either the positive or negative square root of the right side. Since $y(0) = -1$ can occur only if we take the negative square root we have $y = -\sqrt{-2x\cos(x) + 2\sin(x) + 1}$.

(b) Rearranging the equation gives $y' = \frac{y}{x} + e^{y/x}$, so letting $v = y/x$ we have $\frac{dy}{dx} = v + e^v$. Since $y = xv$ the product rule gives $\frac{dy}{dx} = x\frac{dv}{dx} + v$, so the equation becomes

$$x\frac{dv}{dx} + v = v + e^v \implies \frac{dv}{e^v} = \frac{dx}{x} \implies \int \frac{dv}{e^v} = \int \frac{dx}{x} \implies -e^{-v} = \ln|x| + c \implies v = -\ln(-\ln|x| - c),$$

assuming $x \neq 0$. Replacing v with y/x we obtain $y = -x\ln(-\ln|x| - c)$.

□

The *orthogonal trajectories* of a family of curves is a second family of curves, each of which meets the members of the first family at right angles. To calculate the orthogonal trajectories of a family of curves:

(1) take the derivative of a typical member of that family to find its slope m (in terms of x, y and possibly another parameter k);
(2) the slope of an orthogonal trajectory must then be $-\frac{1}{m}$, and we can use the equation we began with to eliminate k, expressing this slope in terms of x and y;
(3) the orthogonal trajectories therefore satisfy the differential equation $\frac{dy}{dx} = -\frac{1}{m}$, which can be solved if it is a separable differential equation.

Example. *Find the orthogonal trajectories of the family of curves* $y = \dfrac{k}{x^2}$.

Solution. Since $\frac{dy}{dx} = -\frac{2k}{x}$ the orthogonal trajectories must have slope $\frac{x}{2k} = \frac{x}{2yx^2} = \frac{1}{2xy}$ (since $yx^2 = k$). Solving this differential equation gives

$$\frac{dy}{dx} = \frac{1}{2xy} \implies \int y\,dy = \int \frac{1}{2x}\,dx \implies \frac{y^2}{2} = \frac{1}{2}\ln|x| + c_1,$$

so the orthogonal trajectories have equations given by $y^2 = \ln|x| + c$. □

A *mixing problem* involves finding the concentration of a solution which is changing in strength. Such problems often lead to separable differential equations.

Example. *A tank contains 1000 litres of brine, with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 litres per minute, and is immediately mixed with the solution; the solution drains from the tank at a rate of 10 litres per minute. How much salt is in the tank at time t minutes?*

Solution. Let $y(t)$ be the salt remaining in the tank after t minutes, so $y(0) = 15$. The concentration at time t is $y(t)/1000$ and $\frac{dy}{dt} = -\frac{y(t)}{1000}10 = -\frac{y(t)}{100}$. Solving this separable equation gives

$$\int \frac{1}{y}\,dy = -\frac{1}{100}\int dt \implies \ln(y) = -\frac{t}{100} + c;$$

since $y(0) = 15$ we have $c = \ln(15)$. We have

$$\ln(y) = \ln(15) - \frac{t}{100} \implies \ln\left(\frac{y}{15}\right) = -\frac{t}{100} \implies \frac{y}{15} = e^{-t/100}.$$

Thus $y = 15e^{-t/100}$. □

Common indefinite integrals

Linearity: $$\int (cf(x) + dg(x))\ dx = c\int f(x)\,dx + d\int g(x)\,dx \quad (c, d \in \mathbb{R})$$

Substitution: $$\int f'\left(g(x)\right) g'(x)\,dx = \int f'(u)\,du = f \circ g(x) \quad u = g(x)$$

Parts: $$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \quad \text{or} \int u\,dv = uv - \int v\,du$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1), \quad \int \frac{1}{x}\,dx = \ln|x| + c$$

Exponential functions: $$\int e^x\,dx = e^x + c, \quad \int b^x\,dx = \frac{b^x}{\ln(b)} + c \quad (b > 0)$$

Trigonometric functions:

$$\int \sin(x)\,dx = -\cos(x) + c, \quad \int \cos(x)\,dx = \sin(x) + c, \quad \int \sec^2(x)\,dx = \tan(x) + c$$

$$\int \csc^2(x)\,dx = -\cot(x) + c, \quad \int \sec(x)\tan(x)\,dx = \sec(x) + c, \quad \int \csc(x)\cot(x)\,dx = -\csc(x) + c$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c, \quad \int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + c$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + c, \quad \int \cot(x)\,dx = \ln|\sin(x)| + c$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c, \quad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c \quad (a > 0)$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + c, \quad \int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + c$$

Formula sheet

Volumes and areas. Sphere of radius r: $V = \frac{4}{3}\pi r^3,\ A = 4\pi r^2$.
Cylinder of radius r and height h: $V = \pi r^2 h,\ A = 2\pi r^2 + 2\pi r h$.
Cone of base radius r and height h: $V = \frac{1}{3}\pi r^2 h,\ A = \pi r\sqrt{r^2+h^2}$.

Equation of a circle and line. The equation of a circle of radius r centred at the point (a, b) is $(x-a)^2 + (y-b)^2 = r^2$.
The slope of a line through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$.
The equation of a line with slope m through the point $P_1(x_1, y_1)$ is $y - y_1 = m(x - x_1)$.

Summations.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Trigonometric identities.

$$1 = \sin^2(x) + \cos^2(x), \quad 1 + \tan^2(x) = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y), \quad \sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y), \quad \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}, \quad \tan(x-y) = \frac{tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x), \quad \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Inverse trigonometric functions.

$$\sin^{-1}(y) = x \iff x = \sin(y) \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\cos^{-1}(y) = x \iff x = \cos(y) \text{ and } 0 \le x \le \pi$$

$$\tan^{-1}(y) = x \iff x = \tan(y) \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Useful trigonometric values.

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Derivatives of elementary functions. We assume that f and g are differentiable functions, and that the various combinations of functions are defined; a, b, c are real numbers.

$$\text{Linearity:}\quad \frac{d}{dx}\left(af(x)+bg(x)\right)=af'(x)+bg'(x)$$

$$\text{Chain rule:}\quad \frac{d}{dx}\left(f\circ g(x)\right)=f'\left(g(x)\right)g'(x)$$

$$\text{Product rule:}\quad \frac{d}{dx}\left((fg)(x)\right)=f(x)g'(x)+f'(x)g(x)$$

$$\text{Quotient rule:}\quad \frac{d}{dx}\left(\frac{f}{g}(x)\right)=\frac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}$$

$$\text{Power rule:}\quad \frac{d}{dx}(x^n)=nx^{n-1},\quad \frac{d}{dx}(c)=0$$

$$\text{Exponential functions:}\quad \frac{d}{dx}\left(e^x\right)=e^x,\quad \frac{d}{dx}\left(b^x\right)=b^x\ln(b)$$

$$\text{Logarithmic functions:}\quad \frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x},\quad \frac{d}{dx}\left(\log_b(x)\right)=\frac{1}{x\ln(b)}$$

Trigonometric functions:

$$\frac{d}{dx}\left(\sin(x)\right)=\cos(x),\quad \frac{d}{dx}\left(\cos(x)\right)=-\sin(x),\quad \frac{d}{dx}\left(\tan(x)\right)=\sec^2(x)$$

$$\frac{d}{dx}\left(\csc(x)\right)=-\csc(x)\cot(x),\quad \frac{d}{dx}\left(\sec(x)\right)=\sec(x)\tan(x),\quad \frac{d}{dx}\left(\cot(x)\right)=-\csc^2(x)$$

Inverse trigonometric functions:

$$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}\left(\cos^{-1}(x)\right)=-\frac{1}{\sqrt{1-x^2}},\quad \frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\csc^{-1}(x)\right)=-\frac{1}{x\sqrt{x^2-1}},\quad \frac{d}{dx}\left(\sec^{-1}(x)\right)=\frac{1}{x\sqrt{x^2-1}},\quad \frac{d}{dx}\left(\cot^{-1}(x)\right)=-\frac{1}{1+x^2}$$

Hooke's Law. The force required to maintain a spring stretched x units beyond its natural length is proportional to x: $f(x)=kx$ for some $k>0$ (provided x is not too large).

Geometric formulas

Our study of integration has allowed us to calculate the area, volume and perimeter of a number of familiar shapes. The required calculations are given in this section.

Circle. A circle of radius r, centred at the origin, has equation $x^2 + y^2 = r^2$; the upper semicircle has equation $y = \sqrt{r^2 - x^2}$.

The area of the circle is therefore twice the area under the curve $y = \sqrt{r^2 - x^2}$:

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = 2\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(\theta)}r\cos(\theta)\,d\theta = 2r^2\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta$$

$$= r^2\int_{-\pi/2}^{\pi/2} 1 + \cos(2\theta)\,d\theta = r^2\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/2}^{\pi/2} = r^2\left(\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)\right) = \pi r^2.$$

The trigonometric substitution is $x = r\sin(\theta)$, so $dx = r\cos(\theta)\,d\theta$ and $x = r \implies \theta = \sin^{-1}(1) = \pi/2$, $x = -r \implies \theta = \sin^{-1}(-1) = -\pi/2$.

The perimeter of the circle is twice the length of the curve $y = \sqrt{r^2 - x^2}$:

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}} \implies 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2},$$

so the perimeter is

$$2\int_{-r}^{r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx = 2r\int_{-\pi/2}^{\pi/2} \frac{r\cos(\theta)}{r\cos(\theta)}\,d\theta = 2r\,[\theta]_{-\pi/2}^{\pi/2} = 2\pi r.$$

The trigonometric substitution is $x = r\sin(\theta)$, so $dx = r\cos(\theta)\,d\theta$ and $x = r \implies \theta = \sin^{-1}(1) = \pi/2$, $x = -r \implies \theta = \sin^{-1}(-1) = -\pi/2$.

Consider a sector of a circle of radius r, with central angle θ ($0 < \theta < \pi/2$). We calculate the area of this sector as the sum of the area of a right-angled triangle with base $r\cos(\theta)$ and height $r\sin(\theta)$ and the area under the curve $y = \sqrt{r^2 - x^2}$ from $x = r\cos(\theta)$ to $x = r$. The area of the triangle is $\frac{1}{2}r^2\sin(\theta)\cos(\theta)$, and

$$\int \sqrt{r^2 - x^2}\,dx = \int_{\alpha} \sin(\alpha)(-r\sin(\alpha))\,d\alpha = -r^2\int \sin^2(\alpha)\,d\alpha = -\frac{r^2}{2}\int 1 - \cos(2\alpha)\,d\alpha$$

$$= -\frac{r^2}{2}(\alpha - \sin(2\alpha)) + c = -\frac{r^2}{2}(\alpha - \sin(\alpha)\cos(\alpha)) + c = -\frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right) + \frac{1}{2}x\sqrt{r^2 - x^2} + c.$$

The trigonometric substitution is $x = r\cos(\theta)$, which works more nicely in this case, and we have used some trigonometric identities. Hence the area under this curve is

$$\left[-\frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right) + \frac{1}{2}x\sqrt{r^2 - x^2}\right]_{r\cos(\theta)}^{r} = \frac{1}{2}\left(0 - (-r^2\theta + r\cos(\theta)\sin(\theta))\right) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin(\theta)\cos(\theta).$$

It follows that the area of the sector in question is

$$\frac{1}{2}r^2\sin(\theta)\cos(\theta) + \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin(\theta)\cos(\theta) = \frac{1}{2}r^2\theta.$$

The area of a sector with central angle larger than $\pi/2$ can be calculated by combining this formula with the formula for the area of a circle.

Ellipse. The longest possible diameter of an ellipse is called its major axis; the minor axis is the diameter of the ellipse which is perpendicular to the major axis. Consider an ellipse centred at the origin, with major axis along the x-axis. Let a denote half the length of the major axis and b half the length of the minor axis, then the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (if $a = b = r$ then this is the equation of a circle of radius r).

The area of the ellipse is therefore twice the area under the curve $y = b\sqrt{1 - \frac{x^2}{a^2}}$:

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx = 2\int_{-a}^{a} b\sqrt{\frac{a^2 - x^2}{a^2}}\,dx = \frac{b}{a}2\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx,$$

which is $\frac{b}{a}$ multiplied by the integral we solved to find the area of a circle of radius a, therefore $A = \frac{b}{a}\pi a^2 = \pi ab$.

The circumference of the ellipse is twice the length of the curve $y = b\sqrt{1 - \frac{x^2}{a^2}}$; the integral given by the arc length formula does not have a solution in terms of elementary functions.

Cylinder. A cylinder of radius r and height h has volume $\pi r^2 h$. The area of the side of the cylinder is $2\pi rh$, since cutting and flattening the surface of the cylinder gives a rectangle with height h and length $2\pi r$. (The circles on the top and bottom both have area πr^2, so some sources give the total surface area of the cylinder as $2\pi r^2 h + 2\pi rh$.)

These expressions for the volume and surface area of a cylinder were taken for granted when we derived the general formulas for volume and surface area, so it is circular to use these formulas in this case.

Cavalieri's principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2 then the volumes of S_1 and S_2 are equal. This principle allows one to calculate the volume of an oblique cylinder (in which the line joining the centres of the top and bottom circles is not perpendicular to these circles).

Cone. To calculate the volume of a cone of base radius r and perpendicular height h we place the centre of the base of the cone at the origin, so the apex of the cone lies on the y-axis at height h. The cone is then the solid formed by rotating the region enclosed by the lines $x = 0$, $y = 0$ and $y = -\frac{h}{r}x + h$ about the y-axis.

To calculate the volume using discs note that a disc centred at the point $(y, 0)$ has radius s, and by similar triangles

$$\frac{s}{h-y} = \frac{r}{h} \implies s = \frac{r(h-y)}{h},$$

so the area of this disc is $A(y) = \pi s^2 = \frac{\pi r^2}{h^2}(h-y)^2$. Therefore the volume of the cone is

$$V = \int_0^h \frac{\pi r^2}{h^2}(h-y)^2\,dy = \frac{\pi r^2}{h^2}\int_0^h h^2 - 2hy + y^2\,dy = \frac{\pi r^2}{h^2}\left[h^2 r - hy^2 + \frac{y^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h.$$

To calculate the volume using shells note that the shell with radius x has height $y = -\frac{h}{r}x + h$, so the volume of the cone is

$$V = \int_0^r 2\pi x\left(-\frac{h}{r}x + h\right)\,dx = 2\pi\int_0^r -\frac{h}{r}x^2 + hx\,dx = 2\pi\left[-\frac{h}{r}\frac{x^3}{3} + \frac{hx^2}{2}\right]_0^r = \frac{1}{3}\pi r^2 h.$$

To find the surface area of a cone one can take a point on the edge of the base and cut the cone along a straight line joining this point to the apex. Flattening this shape gives the sector of a circle with radius l (the length of the line along which the cut was made), arc length $2\pi r$ (since the arc is the circumference of the circle which formed the base of the cone) and angle angle $\theta = \frac{2\pi r}{l}$ (by the definition of a radian). Using the formula for the area of a sector of a circle we get an expression for the surface area:

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\frac{2\pi r}{l} = \pi rl.$$

It would be circular to calculate the surface area of a cone using the formula for area of a surface of revolution because the argument above was used as part of the derivation of this formula.

One can use the above to calculate the volume or surface area of a frustum of a cone by viewing the frustum as a large cone with a smaller cone removed.

Cavalieri's principle allows us to calculate the volume of a slanted cone (in which the line joining the apex to the centre of the base is not perpendicular to the base).

Sphere. To calculate the volume and surface area of a sphere of radius r we position the sphere so its centre is at the origin; the sphere is then the solid obtained by rotating the region under the curve $y = \sqrt{r^2 - x^2}$ about the x-axis.

To calculate the volume using discs note that a disc centred at the point $(x, 0)$ has radius $y = \sqrt{r^2 - x^2}$, so the area of this disc is $A(x) = \pi(r^2 - x^2)$. Therefore the volume of the sphere is

$$V = \int_{-r}^r \pi(r^2 - x^2)\,dx = 2\int_0^r \pi(r^2 - x^2)\,dx = 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3.$$

To calculate the volume using shells note that the shell with radius x has height $2y = 2\sqrt{r^2 - x^2}$. Therefore the volume of the sphere is

$$V = \int_0^r 2\pi x(2\sqrt{r^2 - x^2})\,dx = 4\pi \int_0^r x\sqrt{r^2 - x^2}\,dx = 2\pi \int_0^{r^2} \sqrt{u}\,du = 2\pi \left[\frac{2}{3}u^{3/2}\right]_0^{r^2} = \frac{4}{3}\pi r^3.$$

The substitution is $u = r^2 - x^2$, so $-\frac{1}{2}du = x\,dx$ and $x = 0 \implies u = r^2$, $x = r \implies u = 0$.

The surface area of the sphere is the area of the surface obtained by rotating the curve $y = \sqrt{r^2 - x^2}$ about the x-axis. We have

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}, \quad \text{so} \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}.$$

The surface area of the sphere is therefore

$$\int_{-r}^r 2\pi f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_{-r}^r \sqrt{r^2 - x^2}\frac{r}{\sqrt{r^2 - x^2}}\,dx = 2\pi\,[rx]_{-r}^r = 4\pi r^2.$$

You may notice that some of the above shapes have a special property: the derivative of the formula for their area, with respect to the radius, is the formula for their perimeter; or the derivative of the formula for their volume is the formula for their surface area. For example, the area of a circle of radius r is $A = \pi r^2$, and $\frac{dA}{dr} = 2\pi r$ which is the perimeter of the circle. Similarly, the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$, and $\frac{dV}{dr} = 4\pi r^2$, which is the surface area of the sphere.

To explain these formulas we will show why the integral of the perimeter of a circle with respect to r is the area of a circle; a similar argument shows that the integral of the surface area of a sphere, with respect to the radius, is the volume of the sphere. Place the circle so it is centred at the origin, and divide the x-axis in n intervals $[x_{i-1}, x_i]$ of length $\Delta x = \frac{r}{n}$. A "shell" of the circle is the area of the circle which lies between radius x_{i-1} and x_i, and therefore has area

$$\pi(x_i)^2 - \pi(x_{i-1})^2 = \pi(x_i + x_{i-1})(x_i - x_{i-1}) = \pi(x_i + x_{i-1})\Delta x = 2\pi\overline{x_i}\Delta x,$$

where $\overline{x_i} = \frac{1}{2}(x_i + x_{i-1})$ is the average radius of the shell. The sum of the areas of these shells approximates the area of the circle, and as n increases the approximations become more accurate, so

$$A = \lim_{n\to\infty} \sum_{i=1}^n 2\pi\overline{x_i}\Delta x = \int_0^r 2\pi x\,dx.$$

Thus the area of the circle is the integral of the expression for the perimeter.

It appears that for squares and cubes the perimeter and area (respectively, surface area and volume) are not linked by differentiation: a square of side l has perimeter $4l$ and area $A = l^2$, so $\frac{dA}{dl} = 2l$ is not the perimeter. However, if one defines the "radius" of the square to be $r = l/2$ then the perimeter of the square is $4l = 8r$ and the area of the square is $A = l^2 = (2r)^2 = 4r^2$, so $\frac{dA}{dr} = 8r$ is the perimeter. Similarly, if one defines the "radius" of a cube to be $r = l/2$, where l is the length of one of the edges, then the surface area of the cube is $A = 6l^2 = 24r^2$ and the volume of the cube is $V = l^3 = 8r^3$; once again $\frac{dV}{dr} = A$.

It turns out that any regular shape has the property: the derivative of the area with respect to radius is the perimeter, as long as the radius is defined suitably. The correct notion of radius is the distance from the centre of the object to the midpoint of one of the sides, with the radius and the side meeting at a right-angle.

To prove this relationship we derive a formula for the perimeter and area of a regular polygon with n sides, each side of length l ($n = 3$ is an equilateral triangle, $n = 4$ a square, $n = 5$ a pentagon, *etc.*). The length of the perimeter of this polygon is $P_n = nl$. A radius of the polygon has length r, and joins the centre of the polygon to the midpoint of a side, and forms a right-angled triangle with the line joining the centre to where two sides meet; the angle between the radius and this line is θ. Since the solid is made up of $2n$ such right-angled triangles we have $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$. Also $\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{l/2}{r}$, so $l = 2r\tan(\theta) = 2r\tan(\pi/n)$. The perimeter of the polygon is therefore

$$P_n = nl = 2rn\tan\left(\frac{\pi}{n}\right).$$

The area of the right-angled triangle is half the base multiplied by the height: $\frac{1}{2}\frac{l}{2}r = \frac{1}{2}r^2\tan(\pi/n)$; since the polygon is made up of $2n$ such triangles the area of the polygon is

$$A_n = 2n\frac{1}{2}r^2\tan\left(\frac{\pi}{n}\right) = r^2 n\tan\left(\frac{\pi}{n}\right).$$

Since $n\tan(\pi/n)$ is constant we see that $\frac{dA}{dt} = P$; this formula applies for any regular polygon. Note that when $n = 4$ the above formulas give the correct expressions for the perimeter and area of a square. As the number of sides n goes to infinity our regular polygon approximates a circle, and by l'Hospital's rule

$$\lim_{n\to\infty} n\tan\left(\frac{\pi}{n}\right) = \lim_{t\to 0}\frac{\tan(\pi t)}{t} \overset{H}{=} \lim_{t\to 0}\frac{\pi\sec^2(\pi t)}{1} = \pi,$$

so

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 2rn\tan\left(\frac{\pi}{n}\right), \quad \text{and} \quad \lim_{n\to\infty} A_n = \lim_{n\to\infty} r^2 n\tan\left(\frac{\pi}{n}\right) = \pi r^2,$$

which are the expressions for the perimeter and area of a circle.

Similarly, any regular solid has the property that the surface area of the solid is the derivative of the volume of the solid, with respect to the radius, if the radius of the solid is the distance from the centre of the solid to the centre of any face of the solid. It is possible to use a similar argument to the one given above for perimeter and area to show the relationship between surface area and volume for a regular solid, but it is too long to write out.

A similar principle can be established for solids of rotation if one can define a suitable notion of radius. We will not develop this here, but content ourselves by noting that for a normal cylinder of base radius r the line joining the the centre of the cylinder to the surface, meeting the surface at right angles, is the usual radius r. The derivative of the volume of the cylinder with respect to r is again the expression for the surface area.

The explanations for this phenomenon is always the same: if a suitable notion of radius can be found then the shape can be thought of as nested shells, so the volume of the shape is the integral of the surface area of these shells.