MATH 116 FINAL EXAM REVISION QUESTIONS

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Question 1. *(a) (5 marks) Use the midpoint rule with $n = 10$ to estimate $\int_0^2 \frac{2x^3}{x^2+1}\,dx$. Do not do the arithmetic.*
(b) (4 marks) Calculate the exact value of $\int_0^2 \frac{2x^3}{x^2+1}\,dx$.

Solution. (a) We have $\Delta x = \frac{b-a}{n} = \frac{1}{5}$ and $x_i = i\Delta x = i/5$. (0,0.5,1) Δx and x_i So (0,0.5,1,1.5) formula for trapezoidal rule (0,0.5,1,1.5) correct inputs

$$\begin{aligned}\int_0^2 \frac{2x^3}{x^2+1}\,dx \approx M_{10} &= \Delta x(f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + \cdots + f(\overline{x_9}) + f(\overline{x_{10}}))\\ &= \frac{1}{5}\left(f(1/5) + f(2/5) + f(3/5) + \cdots + f(8/5) + f(9/5) + f(2)\right)\\ &= \frac{1}{5}\left(\frac{2(1/5)^3}{(1/5)^2+1} + \frac{2(2/5)^3}{(2/5)^2+1} + \frac{2(3/5)^3}{(3/5)^2+1} + \cdots + \frac{2(9/5)^3}{(9/5)^2+1} + \frac{2(2)^3}{(2)^2+1}\right).\end{aligned}$$

(0,0.5,1) expression for the estimate

(b) Let $u = x^2$, so $\frac{1}{2}du = x\,dx$, and $x = 0 \implies u = 0$, $x = 2 \implies u = 4$. (0,0.5,1,1.5,2) substitution and partial fractions (0,0.5,1) antiderivative Thus

$$\int_0^2 \frac{2x^3}{x^2+1}\,dx = \int_0^4 \frac{u}{u+1}\,du = \int_0^4 1 - \frac{1}{u+1}\,du = [u - \ln|u+1|]_0^4 = 4 - \ln(5).$$

(0,0.5,1) answer

□

Question 2. *(a) (5 marks) Use the trapezoidal rule with $n = 8$ to estimate $\int_0^{2\pi} \cos^2(x)\,dx$.*
(b) (4 marks) Calculate the exact value of $\int_0^{2\pi} \sin^2(x)\,dx$.

Solution. (a) We have $\delta x = \frac{b-a}{n} = \frac{\pi}{4}$ and $x_i = 0 + i\Delta x$. (0,0.5,1) Δx and x_i So (0,0.5,1) formula for trapezoidal rule (0,0.5,1) correct inputs

$$\begin{aligned}\int_0^{2\pi} \cos^2(x)\,dx \approx T_8 &= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_6) + 2f(x_7) + f(x_8))\\ &= \frac{\pi}{8}\big(\cos^2(0) + 2\cos^2(\pi/4) + 2\cos^2(\pi/2) + 2\cos^2(3\pi/4) + 2\cos^2(\pi)\\ &\qquad + 2\cos^2(5\pi/4) + 2\cos^2(3\pi/2) + 2\cos^2(7\pi/4) + \cos^2(2\pi)\big)\\ &= \frac{\pi}{8}\left(1 + \frac{2}{\sqrt{2}^2} + 0 + \frac{2}{(-\sqrt{2})^2} + 2(-1)^2 + \frac{2}{(-\sqrt{2})^2} + 0 + \frac{2}{\sqrt{2}^2} + 1\right)\\ &= \pi.\end{aligned}$$

(0,0.5,1) evaluating trig functions (0,0.5,1) answer

(b) Using the identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ we have (0,0.5,1,1.5) trig identity (0,0.5,1,1.5) antiderivative

$$\int_0^{2\pi} \cos^2(x)\,dx = \frac{1}{2}\int_0^{2\pi} 1 + \cos(2x)\,dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_0^{2\pi} = \frac{1}{2}\left((2\pi + 0) - 0\right) = \pi.$$

(0,0.5,1) answer

□

Question 3. *Evaluate the following integrals.*

(a) *(5 marks)* $\int t\csc^2(t)\,dt$

(b) *(5 marks)* $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

(c) *(5 marks)* $\int \frac{1}{x^2-ax}\,dx \;\; (a \neq 0)$

(d) *(5 marks)* $\int \tan^5(x)\sec^3(x)\,dx$

(e) *(5 marks)* $\int \frac{x^3+4x+3}{(x^2+1)(x^2+4)}\,dx$

(f) *(5 marks)* $\int e^{-\theta}\cos(2\theta)\,d\theta$

(g) *(5 marks)* $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$

(h) *(5 marks)* $\int \sin(8x)\cos(5x)\,dx$

(i) *(5 marks)* $\int_{-1}^{0} x\sqrt{1-x^2}\,dx$

Solution. (a) Let $u = t$, $dv = \csc^2(t)\,dt$. Then $du = dt$ and $v = -\cot(t)$. (0,0.5,1) components of integration by parts By integration by parts we have (0,0.5,1) using integration by parts formula

$$\begin{aligned}\int t\csc^2(t)\,dt &= -t\cot(t) - \int -\cot(t)\,dt \\ &= -t\cot(t) + \int \frac{\cos(t)}{\sin(t)}\,dt\end{aligned}$$

To find the integral in the last term substitute $z = \sin(t)$, so $dz = \cos(t)\,dt$. (0,0.5,1) substitution Therefore

$$\begin{aligned}\int t\csc^2(t)\,dt &= -t\cot(t) + \int \frac{\cos(t)}{\sin(t)}\,dt \\ &= -t\cot(t) + \int \frac{1}{z}\,dz \\ &= -t\cot(t) + \ln|z| + c \quad \text{(0,0.5,1) integral from substitution} \\ &= -t\cot(t) + \ln|\sin(t)| + c. \quad \text{(0,0.5,1) answer}\end{aligned}$$

(b) Let $x = 3\sin(\theta)$, where $-\pi/2 \leq \theta \leq \pi/2$, so $dx = 3\cos(\theta)\,d\theta$. (0,0.5,1) choice of trig sub Then (0,0.5,1)

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2(\theta)} = \sqrt{9\cos^2(\theta)} = 3|\cos(\theta)| = 3\cos(\theta).$$

Thus

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{9\sin^2(\theta)}{3\cos(\theta)}3\cos(\theta)\,d\theta \\ &= 9\int \sin^2(\theta)\,d\theta \\ &= \frac{9}{2}\int 1-\cos(2\theta)\,d\theta \quad \text{(0,0.5,1) use of identity} \\ &= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + c \\ &= \frac{9}{2}\theta - \frac{9}{4}2\sin(\theta)\cos(\theta) + c. \quad \text{(0,0.5) antiderivative}\end{aligned}$$

Since $\sin(\theta) = x/3$ we have $\cos(\theta) = \sqrt{9-x^2}/3$ (by drawing a right-angled triangle), (0,0.5) so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \frac{9}{2}\theta - \frac{9}{4}2\sin(\theta)\cos(\theta) + c \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + c \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + c. \quad \text{(0,0.5,1) substitute for } \theta \text{ to answer}
\end{aligned}$$

(c) We have (0,0.5,1,1.5) form of partial fractions

$$\frac{1}{x(x-a)} = \frac{A}{x} + \frac{B}{x-a} \iff 1 = A(x-a) + Bx,$$

which solves to give $A = -1/a$ and $B = 1/a$ (first substitute $x = a$, then $x = 0$). (0,0.5,1) find A and B by any method Therefore

$$\begin{aligned}
\int \frac{1}{x(x-a)}\,dx &= \int -\frac{1}{ax} + \frac{1}{a(x-a)}\,dx \quad \text{(0,0.5,1) simplify integral with partial fractions} \\
&= -\frac{1}{a}\ln|x| + \frac{1}{a}\ln|x-a| + c. \quad \text{(0,0.5,1,1.5) antiderivative}
\end{aligned}$$

(d) We use the identity $\tan^2(x) = \sec^2(x) - 1$ to eliminate all but one of the $\tan(x)$ factors, then substitute $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$. So (0,0.5,1) approach for powers of sec and tan

$$\begin{aligned}
\int \tan^5(x)\sec^3(x)\,dx &= \int \tan^4(x)\sec^2(x)\sec(x)\tan(x)\,dx \\
&= \int \left(\sec^2(x) - 1\right)^2 \sec^2(x)\sec(x)\tan(x)\,dx \quad \text{(0,0.5,1) use of this identity} \\
&= \int (u^2-1)^2 u^2\,du \quad \text{(0,0.5,1) substitution} \\
&= \int u^6 - 2u^4 + u^2\,du \\
&= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + c \quad \text{(0,0.5,1) antiderivative} \\
&= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + c. \quad \text{(0,0.5,1) answer}
\end{aligned}$$

(e) We have (0,0.5,1,1.5) form of partial fractions

$$\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \iff 4y^2 - 7y - 12 = A(y+2)(y-3) + By(y-3) + Cy(y+2),$$

which solves to give $A = 2$, $B = 9/5$, $C = 1/5$ (first substitute $y = 0$, then $y = -2$, then $y = 3$). (0,0.5,1,1.5) find A, B and C by any method Therefore

$$\begin{aligned}
\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy &= \int_1^2 \frac{2}{y} + \frac{9}{5}\frac{1}{y+2} + \frac{1}{5}\frac{y-3}{}dy \\
&= \left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3|\right]_1^2 \quad \text{(0,0.5,1) antiderivative} \\
&= 2\ln(2) + \frac{9}{5}\ln(4) + \frac{1}{5}\ln(1) - 2\ln(1) - \frac{9}{5}\ln(3) - \frac{1}{5}\ln(2) \\
&= \frac{9}{5}(3\ln(2) - \ln(3)). \quad \text{(0,0.5,1) evaluate to answer}
\end{aligned}$$

Other forms of the final answer, such as $\frac{9}{5}\ln(8/3)$, are acceptable.

(f) Let $u = e^{-\theta}$, $dv = \cos(2\theta)\, d\theta$. Then $du = -e^{-\theta}$ and $v = \frac{1}{2}\sin(2\theta)$. By integration by parts

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)\, d\theta &= \frac{1}{2}e^{-\theta}\sin(2\theta) - \int \frac{1}{2}\sin(2\theta)(-e^{-\theta})\, d\theta \\ &= \frac{1}{2}e^{-\theta}\sin(2\theta) + \frac{1}{2}\int e^{-\theta}\sin(2\theta)\, d\theta.\end{aligned}$$

We use integration by parts again to find the integral on the right side. (0,0.5) recognise need second integration by parts Let $s = e^{-\theta}$, $dt = \sin(2\theta)\, d\theta$. Then $ds = -e^{-\theta}$ and $v = \frac{1}{2}\cos(2\theta)$. By integration by parts (0,0.5,1) second integration by parts

$$\int e^{-\theta}\sin(2\theta)\, d\theta = -\frac{1}{2}e^{-\theta}\cos(2\theta) - \frac{1}{2}\int e^{-\theta}\cos(2\theta)\, d\theta.$$

Hence (0,0.5,1) formula involving two instances of the integral

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)\, d\theta &= \frac{1}{2}e^{-\theta}\sin(2\theta) + \frac{1}{2}\int e^{-\theta}\sin(2\theta)\, d\theta \\ &= \frac{1}{2}e^{-\theta}\sin(2\theta) + \frac{1}{2}\left(-\frac{1}{2}e^{-\theta}\cos(2\theta) - \frac{1}{2}\int e^{-\theta}\cos(2\theta)\, d\theta\right) \\ &= \frac{1}{2}e^{-\theta}\sin(2\theta) - \frac{1}{4}e^{-\theta}\cos(2\theta) - \frac{1}{4}\int e^{-\theta}\cos(2\theta)\, d\theta.\end{aligned}$$

Therefore

$$\frac{5}{4}\int e^{-\theta}\cos(2\theta)\, d\theta = \frac{1}{2}e^{-\theta}\sin(2\theta) - \frac{1}{4}e^{-\theta}\cos(2\theta),$$

so in general (0,0.5,1) solve to answer

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)\, d\theta &= \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin(2\theta) - \frac{1}{4}e^{-\theta}\cos(2\theta)\right) \\ &= \frac{2}{5}e^{-\theta}\sin(2\theta) - \frac{1}{5}e^{-\theta}\cos(2\theta) + c.\end{aligned}$$

(g) We have (0,0.5,1) form of partial fractions

$$\frac{x^3+4x+3}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \iff x^3+4x+3 = (A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D),$$

which solves to give $A = 1$, $B = 1$, $C = 0$, $D = -1$ (equate coefficients). (0,0.5,1,1.5,2) find A, B and C by any method Thus

$$\begin{aligned}\int \frac{x^3+4x+3}{(x^2+1)(x^2+4)}\, dx &= \int \frac{x+1}{x^2+1} - \frac{1}{x^2+4}\, dx \\ &= \int \frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x^2+4}\, dx.\end{aligned}$$

For the first term we use the mental substitution $u = x^2$; the second and third are integrals of the form $\int (x^2+a^2)^{-1}\, dx$, which give $\frac{1}{a}\tan^{-1}(x/a) + c$, so we obtain

$$\begin{aligned}\int \frac{x^3+4x+3}{(x^2+1)(x^2+4)}\, dx &= \int \frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x^2+4}\, dx \\ &= \frac{1}{2}\ln(x^2+1) + \tan^{-1}(x) - \frac{1}{2}\tan^{-1}\left(\frac{x}{a}\right) + c.\end{aligned}$$ (0,0.5,1,1.5,2) antiderivatives

(h) Note that $x^2+2x+5 = (x+1)^2+4$. Let $x+1 = 2\tan(\theta)$ where $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2(\theta)\, d\theta$. (0,0.5,1) trig sub Then (0,0.5,1)

$$\sqrt{x^2+2x+5} = \sqrt{(x+1)^2+4} = \sqrt{4\tan^2(\theta)+4} = \sqrt{4\sec^2(\theta)} = 2\sec(\theta),$$

the last identity following from the range of values allowed for θ. Thus

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+5}}\,dx &= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,d\theta \\ &= \int \sec^2(\theta)\,d\theta \\ &= \ln|\sec(\theta)+\tan(\theta)|+c \quad \text{(0,0.5,1) antiderivative} \\ &= \ln\left|\frac{\sqrt{x^2+2x+5}}{2}+\frac{x+1}{2}\right|+c \quad \text{(0,0.5,1) substitute to answer} \\ &= \ln\left|\sqrt{x^2+2x+5}+x+1\right|+c_1,\end{aligned}$$

with $c_1 = c - \ln(2)$. The expressions for $\sec(\theta)$ and $\tan(\theta)$ in terms of x are found by drawing a right angled triangle. (0,0.5)

(i) Using the identity $\sin(A)\cos(B) = \frac{1}{2}(\sin(A-B)+\sin(A+B))$ (0,0.5,1) using a suitable identity (0,0.5,1,1.5) correct use of identity

$$\begin{aligned}\int \sin(8x)\cos(5x)\,dx &= \int \frac{1}{2}\left(\sin(8x-5x)+\sin(8x+5x)\right)\,dx \\ &= \frac{1}{2}\int \sin(3x)+\sin(13x)\,dx \\ &= \frac{1}{2}\left(-\frac{1}{3}\cos(3x)-\frac{1}{13}\cos(13x)\right)+c \quad \text{(0,0.5,1,1.5) antiderivative} \\ &= -\frac{1}{6}\cos(3x)-\frac{1}{26}\cos(13x)+c. \quad \text{(0,0.5,1) answer}\end{aligned}$$

(j) Let $u = 1-x^2$, so $du = -2xdx$ and $xdx = -\frac{1}{2}du$; $x = 0 \iff u = 1$ and $x = -1 \iff u = 0$. (0,0.5,1) valid substitution Therefore

$$\begin{aligned}\int_{-1}^{0} x\sqrt{1-x^2}\,dx &= -\frac{1}{2}\int_0^1 \sqrt{u}\,du \quad \text{(0,0.5,1) substitution} \\ &= \left[-\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}\right]_0^1 \quad \text{(0,0.5,1) antiderivative} \\ &= -\frac{1}{3}(1)-0 \quad \text{(0,0.5,1) evaluate} \\ &= -\frac{1}{3} \quad \text{(0,0.5,1) answer.}\end{aligned}$$

□

Question 4. *(7 marks) A cable 60 feet long, weighing 180 lbs, hangs from a winch on a crane. How much work is done if the winch winds in 25 feet of cable?*

Solution. Once x feet of cable have been wound in by the winch there remains $60-x$ feet of cable hanging. The hanging portion of the cable weighs $3(60-x)$ lbs, (0,0.5,1,1.5,2) weight function and requires $3(60-x)\Delta x$ ft-lbs of work to be lifted Δx feet. (0,0.5,1) expression for work on segment Therefore the work required is marking(0,0.5,1,1.5,2) set up integral

$$W = \int_0^{25} 3(60-x)\,dx = \int_0^{25} 180-3x\,dx = \left[180x - \frac{3x^2}{2}\right]_0^{25} = 3562.5.$$

(0,0.5,1,1.5,2) integrate to answer

Alternative solution: the part of the cable hanging down from x to $x+\Delta x$ feet weighs $3\Delta x$ lbs and must be lifted D_x feet, so the work required to lift this part of the cable is $3D_x\Delta x$ ft-lbs. (0,0.5,1,1.5) expression for work The part of the cable between $x = 0$ and $x = 25$ must be lifted $D_x = x$ feet, (0,0.5,1) first distance while the part of the cable between $x = 25$ and $x = 60$ must be lifted $D_x = 25$ feet. (0,0.5,1) second distance

Hence the total work involved is (0,0.5,1) expression for total work (0,0.5,1) split integral

$$W = \int_0^{60} 3D_x \, dx = \int_0^{25} 3x \, dx + \int_{25}^{60} 3(25) \, dx = \left[\frac{3x^2}{2}\right]_0^{25} + [3(25)x]_{25}^{60} = 937.5 + 2625 = 3562.5.$$

(0,0.5,1,1.5) integrate to answer □

Question 5. *(7 marks) A chain of length 10 metres, mass 80 kg, lies on the ground. How much work is done in raising one end of the chain to a height of 6 metres, so that the chain forms an L-shape? You may assume that the friction between the chain and the ground is negligible, and that the mass of the chain is evenly distributed along its length. Give your answer in terms of the acceleration due to gravity g.*

Solution. The part of the chain x metres from the end which is lifted is raised a distance $6 - x$ metres when $0 \le x \le 6$ and 0 metres if $6 < x \le 10$. (0,0.5,1,1.5,2) reasoning The chain has mass 8 kilograms per metre, so (using the assumption) the part of the chain from x to $x + \Delta x$ metres from the lifted end weighs $8g\Delta x$ N, where g is the acceleration due to gravity. (0,0.5,1,1.5) weight of segment The work required is (0,0.5,1,1.5,2) setting up integral

$$W = \int_0^6 8g(6-x) \, dx = 8g\left[6x - \frac{x^2}{2}\right]_0^6 = 8g(36-18) = 144g.$$

(0,0.5,1,1.5) integrate to answer □

Question 6. *Consider the region R bounded by the curves $y = x^2$ and $y = cx$ $(c > 0)$.*

(a) (5 marks) Find the area of the region R.
(b) (6 marks) Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region R about the x-axis.
(c) (6 marks) Use the disc/washer method to find the volume of the solid obtained by rotating the region R about the line $x = -c$.

Solution. (a) The curves meet when $x^2 = cx \iff x(x-c) = 0 \iff x = 0$ or $x = c$. (0,0.5,1) intersection points Testing with $x = c/2 \in [0, c]$ we see $cx = c^2/2 \ge c^2/4 = x^2$, so the top curve is $y = cx$ and the bottom curve is $y = x^2$. (0,0.5,1) correctly identifying top and bottom The area is (0,0.5,1,1.5) setting up integral correctly

$$A = \int_0^c cx - x^2 \, dx = \left[\frac{cx^2}{2} - \frac{x^3}{3}\right]_0^c = \frac{c^3}{2} - \frac{c^3}{2} = \frac{c^3}{6}.$$

(0,0.5,1,1.5) evaluate integral to answer

(b) Since the axis of rotation is $y = 0$ we will integrate with respect to y. The radius of a typical shell is y, the height of a shell is $\sqrt{y} - \frac{y}{c}$. (0,0.5,1) radius and height in terms of y Therefore the required volume is (0,0.5,1) formula for shells (0,0.5,1,1.5,2) setting up integral correctly

$$\begin{aligned} V = \int_a^b 2\pi r h \, dy &= 2\pi \int_0^{c^2} y\left(\sqrt{y} - \frac{y}{c}\right) dy \\ &= 2\pi \int_0^{c^2} y^{3/2} - \frac{y^2}{c} \, dy \\ &= 2\pi \left[\frac{2y^{5/2}}{5} - \frac{y^3}{3c}\right]_0^{c^2} \qquad \text{(0,0.5,1) antiderivative} \\ &= 2\pi\left(\frac{2c^5}{5} - \frac{c^6}{3c}\right) \\ &= 2\pi c^5\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2\pi}{15}c^5 \qquad \text{(0,0.5,1) evaluate to answer.} \end{aligned}$$

(c) Again our integral is with respect to y. The inner radius of a washer is $c+y/c$, the outer radius is $c+\sqrt{y}$. Therefore the area function is (0,0.5,1,1.5) area function

$$A(y) = \pi\left(c+\sqrt{(y)}\right)^2 - \pi\left(c+\frac{y}{c}\right)^2 = \pi\left(c^2+2c\sqrt{y}+y-\left(c^2+2y+\frac{y^2}{c^2}\right)\right)$$
$$= \pi\left(2cy^{1/2}-y-\frac{y^2}{c^2}\right).$$

Therefore the required volume is (0,0.5,1) formula for shells (0,0.5,1,1.5) setting up integral correctly

$$V = \int_a^b A(y)\,dy = \pi\int_0^{c^2} 2cy^{1/2}-y-\frac{y^2}{c^2}\,dy$$
$$= \pi\left[\frac{4cy^{3/2}}{3}-\frac{y^2}{2}-\frac{y^3}{3c^2}\right]_0^{c^2} \quad \text{(0,0.5,1) antiderivative}$$
$$= \pi\left(\frac{4c^4}{3}-\frac{c^4}{2}-\frac{c^4}{3}\right)$$
$$= \frac{\pi c^4}{2} \quad \text{(0,0.5,1) evaluate to answer.}$$

□

Question 7. *Consider the region R bounded by the curves $y=\ln(x)$, $y=0$ and $x=2$.*

(a) (5 marks) Find the area of the region R.

(b) (6 marks) Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region R about the y-axis.

(c) (6 marks) Use the disc/washer method to find the volume of the solid obtained by rotating the region R about the x-axis.

Solution. (a) The curves $y=0$ and $y=\ln(x)$ intersect when $\ln(x)=0 \iff x=1$. (0,0.5,1) intersection points The upper curve is $y=\ln(x)$, so the required area is (0,0.5,1) setting up integral

$$A = \int_1^2 \ln(x)\,dx = [x\ln(x)]_1^2 - \int_1^2 1\,dx = [x\ln(x)]_1^2 - [x]_1^2$$
$$= (2\ln(2)) - (2-1) = 2\ln(2)-1,$$

using integration by parts with $u=\ln(x) \implies du=\frac{1}{x}\,dx$ and $dv=dx \implies v=x$. (0,0.5,1,1.5,2) integration by parts (0,0.5,1) answer

(b) The axis of rotation is $x=0$ so we integrate with respect to x. The radius of a shell is x and the height of a shell is $\ln(x)$. (0,0.5,1) radius and height Therefore the required volume is (0,0.5,1,1.5,2) setting up integral correctly

$$V = \int_1^2 2\pi rh\,dx = 2\pi\int_0^2 x\ln(x)\,dx$$
$$= 2\pi\left(\left[\frac{x^2}{2}\ln(x)\right]_1^2 - \int_1^2 \frac{x}{2}\,dx\right) \quad \text{parts: } u=\ln(x),\ dv=x\,dx$$
$$= 2\pi\left(\left[\frac{x^2}{2}\ln(x)\right]_1^2 - \left[\frac{x^2}{4}\right]_1^2\right) \quad \text{(0,0.5,1,1.5,2) integrate by parts}$$
$$= 2\pi\left((2\ln(2)-0)-\left(1-\frac{1}{4}\right)\right)$$
$$= 2\pi\left(2\ln(2)-\frac{3}{4}\right). \quad \text{(0,0.5,1) evaluate to answer}$$

(c) The axis of rotation is $y=0$ so we integrate with respect to x. The radius of a typical disc is $y=\ln(x)$, so the area function is (0,0.5,1) area function

$$A(x) = \pi\left(\ln(x)\right)^2.$$

The required volume is therefore (0,0.5,1) formula for shells (0,0.5,1,1.5) setting up integral correctly

$$\begin{aligned} V = \int_1^2 A(x)\,dx &= \pi \int_1^2 (\ln(x))^2\,dx \\ &= \pi\left(\left[x(\ln(x))^2\right]_1^2 - 2\int_1^2 \ln(x)\,dx\right) \quad \text{parts: } u=(\ln(x))^2,\ dv = x\,dx \\ &= \pi\left(\left[x(\ln(x))^2\right]_1^2 - 2\left[x\ln(x) - x\right]_1^2\right) \quad \text{parts as in (a)} \\ &= \pi\left(\left(2(\ln(2))^2 - 0\right) - 2\left((2\ln(2)-2)-(-1)\right)\right) \\ &= \pi\left(2(\ln(2))^2 - 4\ln(2) + 2\right). \end{aligned}$$

(0,0.5,1,1.5,2) integration by parts (0,0.5) answer

□

Question 8. *(4 marks) Calculate* $\lim_{x\to 0} \frac{x-\sin(x)}{x-\tan(x)}$.

Solution. The limit is of type $0/0$. (0,0.5) Applying l'Hospital's rule gives (0,0.5,1) use of l'Hospital

$$\lim_{x\to 0}\frac{x-\sin(x)}{x-\tan(x)} = \lim_{x\to 0}\frac{1-\cos(x)}{1-\sec^2(x)},$$

which is again of type $0/0$. (0,0.5) recognising result Applying l'Hospital's rule again gives (0,0.5,1) second application (0,0.5,1) answer

$$\begin{aligned} \lim_{x\to 0}\frac{x-\sin(x)}{x-\tan(x)} = \lim_{x\to 0}\frac{1-\cos(x)}{1-\sec^2(x)} &= \lim_{x\to 0}\frac{-(-\sin(x))}{-2\sec(x)\sec(x)\tan(x)} \\ &= \lim_{x\to 0} -\frac{1}{2}\cos^3(x) \\ &= -\frac{1}{2}(1)^3 = -\frac{1}{2}. \end{aligned}$$

□

Question 9. *(4 marks) Calculate* $\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}$.

Solution. The limit is of type $\frac{\infty}{\infty}$. (0,0.5) Applying l'Hospital's rule gives (0,0.5,1,1.5) applying rule

$$\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})},$$

which is of type $\frac{0}{0}$. (0,0.5) We calculate this limit by rearranging: (0,0.5,1,1.5) rearrange to answer

$$\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

Alternatively the indeterminate form of type $0/0$ can be found by a second application of l'Hospital's rule. □

Question 10. *(5 marks) Determine whether the improper integral is divergent or convergent. If it is convergent find its value.* $\int_0^4 \frac{1}{(x-2)(x+1)}\,dx$

Solution. The integrand has an infinite discontinuity at $x=2$. First we must calculate $\int_0^2 \frac{1}{(x-2)(x+1)}\,dx$. By definition (0,0.5,1) definition

$$\int_0^2 \frac{1}{(x-2)(x+1)}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)(x+1)}\,dx.$$

To calculate this integral we use partial fractions:

$$\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \iff 1 = A(x+1) + B(x-2);$$

taking $x = 2$ gives $A = 1/3$ and taking $x = -1$ gives $B = -1/3$. (0,0.5,1,1.5) partial fractions Hence (0,0.5,1) calculating integral

$$\begin{aligned}
\int_0^2 \frac{1}{(x-2)(x+1)}\,dx &= \lim_{t\to 2^-} \int_0^t \frac{1}{(x-2)(x+1)}\,dx \\
&= \lim_{t\to 2^-} \int_0^2 -\frac{\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x+1}\,dx \\
&= \lim_{t\to 2^-} \left[\frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+1|\right]_0^t \\
&= \lim_{t\to 2^-} \left(\frac{1}{3}\ln|t-2| - \frac{1}{3}\ln|t+1| - \frac{1}{3}\ln(2) + 0\right).
\end{aligned}$$

This limit is $-\infty$, since $\ln|t-2| \to -\infty$ as $t \to 2^-$. (0,0.5,1) limit with reasoning (0,0.5) divergent □

Question 11. *(a) (4 marks) Show that the improper integral* $\int_0^\infty \frac{2}{e^x}\,dx$ *is convergent by determining its value.*

(b) (2 marks) State the comparison theorem for improper integrals of type 1.

(c) (5 marks) Use parts (a) and (b) to show that the improper integral $\int_0^\infty \frac{\tan^{-1}(x)}{2+e^x}\,dx$ *is convergent.*

Solution. (a) By definition (0,0.5,1,1.5) definition (0,0.5,1) calculate integral (0,0.5,1) calculate limit (0,0.5) correct value

$$\int_0^\infty \frac{2}{e^x}\,dx = \lim_{t\to\infty} \int_0^t \frac{2}{e^x}\,dx = \lim_{t\to\infty} \left[-\frac{2}{e^x}\right]_0^t = \lim_{t\to\infty} 2 - \frac{2}{e^t} = 2.$$

(b) Comparison theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then if $\int_a^\infty f(x)\,dx$ is convergent the integral $\int_a^\infty g(x)\,dx$ is also convergent; if $\int_a^\infty g(x)\,dx$ is divergent the integral $\int_a^\infty f(x)\,dx$ is also divergent. (0,0.5,1,1.5,2)

(c) Since $x > 0$ and $\tan^{-1}(x) < \frac{\pi}{2} < 2$ we have (0,0.5,1,1.5,2) inequality for comparison

$$\frac{\tan^{-1}(x)}{2+e^x} < \frac{2}{2+e^x} < \frac{2}{e^x}.$$

By (a) $\int_0^\infty \frac{2}{e^x}\,dx$ is convergent, (0,0.5,1) use of (a) so by the comparison theorem (0,0.5,1,1.5,2) use of comparison theorem to answer $\int_0^\infty \frac{\tan^{-1}(x)}{2+e^x}\,dx$ is also convergent.

□

Question 12. *(6 marks) Find the length of the curve* $y = 1 + 6x^{\frac{3}{2}}$, $0 \leq x \leq 1$.

Solution. We have $\frac{dy}{dx} = 9x^{1/2}$. (0,0.5,1) Using the arc length formula (0,0.5,1,1.5) correct formula (0,0.5,1,1.5) rearranging integrand suitably

$$\begin{aligned}
L = \int_0^1 \sqrt{1 + \left(9x^{\frac{1}{2}}\right)^2}\,dx &= \int_0^1 \sqrt{1+81x}\,dx \\
&= \int_1^{82} \sqrt{u}\frac{1}{81}\,du \quad \text{(0,0.5,1) substitution} \\
&= \frac{1}{81}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{82} \quad \text{(0,0.5,1) antiderivative} \\
&= \frac{2}{243}(82\sqrt{82} - 1). \quad \text{(0,0.5,1) evaluate to answer}
\end{aligned}$$

□

Question 13. *(6 marks) Find the length of the curve* $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 2$.

Solution. We have $\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$. (0,0.5,1) Using the arc length formula (0,0.5,1) correct formula (0,0.5,1) rearranging integrand suitably

$$\begin{aligned} L = \int_1^2 \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\, dx &= \int_1^2 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx \\ &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx \\ &= \int_1^2 \left| x^2 + \frac{1}{4x^2} \right| dx \\ &= \int_1^2 x^2 + \frac{1}{4x^2}\, dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 \quad \text{(0,0.5,1) antiderivative} \\ &= \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{59}{24}. \quad \text{(0,0.5,1) evaluate to answer} \end{aligned}$$

□

Question 14. *(6 marks) Find the area of the surface obtained by rotating the curve $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$, about the x-axis.*

Proof. We have (0,0.5,1) derivative

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}.$$

Hence (0,0.5,1) simplifying

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2}{1-x^2} - \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

The required surface area is (0,0.5,1.5) formula for area (0,0.5,1,1.5) setting up integral

$$S = \int_{-1}^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{-1}^1 \sqrt{1-x^2} \frac{1}{\sqrt{1-x^2}}\, dx = 2\pi \left[x\right]_{-1}^1 = 4\pi.$$

(0,0.5,1) integrate to answer □

Question 15. *(6 marks) Find the area of the surface obtained by rotating the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, $1 \leq x \leq 2$, about the y-axis.*

Proof. We have (0,0.5,1) derivative

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}.$$

Hence (0,0.5,1) simplifying

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

The required surface area is (0,0.5,1.5) formula for area (0,0.5,1,1.5) setting up integral

$$\begin{aligned}
S = \int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx &= 2\pi\int_1^2 \left(\frac{x}{2}+\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx \\
&= \pi\int_1^2 x^2+1\,dx \\
&= \pi\left[\frac{x^3}{3}+x\right]_1^2 \\
&= \pi\left(\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)\right) \\
&= \frac{10}{3}\pi.
\end{aligned}$$

(0,0.5,1) integrate to answer □

Question 16. *(6 marks) A curve passes through the point $(0,3)$ and has the property that the slope of the curve at the point P and twice the y-coordinate of P sum to zero. Find the equation of the curve.*

Proof. Suppose the equation of the curve is $y=f(x)$, we are told that $f'(x)+2f(x)=0 \iff f'(x) = -2f(x)$. (0,0.5,1,1.5) write the given equation We know that the only solution to this differential equation is $y = Ce^{kx}$. (0,0.5,1,1.5,2) form of solution Therefore $3 = y(0) = Ce^0 = C$. (0,0.5,1) find C To find k note that $y' = 3ke^{kx}$, (0,0.5) derivative so the differential equation implies $3ke^{kx} = -2(3e^{kx})$, so $k=-2$. (0,0.5,1) find k Therefore the equation is $y = 3e^{-2x}$. □

Question 17. *(6 marks) Find a solution of the differential equation which satisfies the given initial condition. Write your answer in the form $y = f(x)$. $\dfrac{dy}{dx} = xe^y$, $y(0)=0$.*

Proof. The separable equation gives (0,0.5,1,1.5,2) separating and integrating (0,0.5,1) integrals

$$e^{-y}\,dy = x\,dx \implies \int e^{-y}\,dy = \int x\,dx \implies -e^{-y} = \frac{1}{2}x^2+c.$$

Using $y(0)=0$ gives $-e^0 = 0+c$, so $c=-1$. (0,0.5,1,1.5) find constant Hence (0,0.5,1,1.5) algebra to answer

$$e^{-y} = -\frac{1}{2}x^2+1 \implies -y = \ln\left(1-\frac{1}{2}x^2\right) \implies y = -\ln\left(1-\frac{1}{2}x^2\right).$$

□

Question 18. *(6 marks) Find the orthogonal trajectories of the family of curves $y = \dfrac{k}{x}$.*

Proof. Differentiating: (0,0.5,1) derivative

$$\frac{d}{dx}y = \frac{d}{dx}\frac{k}{x} \implies y' = -\frac{k}{x^2}.$$

Substituting $k = xy$ (0,0.5,1) this substitution gives $y' = -\frac{xy}{x^2} = -\frac{y}{x}$. Therefore the orthogonal trajectories must satisfy $\frac{dy}{dx} = \frac{x}{y}$. (0,0.5,1) correct equation Solving the separable differential equation gives (0,0.5,1,1.5,2) solving separable equation

$$\int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2+c_1 \implies x^2-y^2 = c.$$

(0,0.5,1) answer □