MATH 110: CALCULUS I REVISION SHEET

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This page is a checklist of things you should know for the exam. Do not think that it contains everything which you might be asked on the exam, or that there are any hints about what the exam questions will be hidden in here. The idea is to summarise the main points.

Make sure you know the notation for sets of numbers and inequalities, particularly intervals : $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, *etc.*

You should be familiar with functions and their properties.

- A function $f : A \to B$ is a rule, which takes as *input* a number $x \in A$ and gives an output $f(x)$ in B. The set of numbers A is called the *domain* of f.
- Functions can only give one output for any input; a curve is the graph of a function if it passes the *vertical line test*. Make sure you understand the difference between inputs and outputs!
- A function f is called *odd* if $f(-x) = -f(x)$ for all x in the domain of f such that $-x$ is also in the domain of f; f is called *even* if $f(-x) = f(x)$ for all such x.
- Given functions f and g we can form new functions cf ($c \in \mathbb{R}$), $f + g$, fg, $\frac{f}{g}$, $f \circ g$. You should know how to recognise each of these new functions built from simpler ones, and how to find the domain of each.
- A function f is called *one-to-one* if $a \neq b$ implies $f(a) \neq f(b)$ (equivalently $f(a) = f(b)$ implies $a = b$); one-to-one functions pass the *horizontal line test*. If f is a one-to-one function we can find the *inverse function* f^{-1}, which satisfies $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$.
- The *graph* of a function $f : A \to B$ is the set of points $\{(x, f(x)) : x \in A\}$. You should be able to recognise the properties above, as well as the functions below, from graphs.

The functions we have seen include:

- polynomial functions $f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$;
- power functions involving rational numbers: $x^{\frac{p}{q}} = \sqrt[q]{x^p}$ and $x^{-m} = \frac{1}{x^m}$;
- rational functions — quotients of two polynomial functions;
- exponential functions $f(x) = b^x$ and their inverses the logarithmic functions $g(y) = \log_b(y)$, including those with the special base e, and the laws which they obey;
- trigonometric functions and their inverses, and their interpretation as the horizontal and vertical distances of a point on the plane, and the definition of and reasons for using radians;
- the absolute value function $f(x) = |x|$;
- piecewise defined functions;

- combinations of all the above types of functions.

You should understand the definition of the domain of a function well enough to be able to figure out the domains of all the above functions, though there is no need to learn the domains of all these functions.

You should also remember the Pythagorean theorem: $h^2 = a^2 + b^2$, where h is the hypotenuse of a right angled triangle and a and b are the other two sides. The Pythagorean theorem leads to the first trigonometric identity $\sin^2(x) + \cos^2(x) = 1$, which you should remember. It might be helpful to know how other trigonometric identities can be derived from this one, but you are not required to learn all the trig identities.

The methods for finding the equation of a line are useful, particularly when derivatives are interpreted as tangent lines. The slope of a line through points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is $m = \frac{y_1 - y_0}{x_1 - x_0}$. The equation of a line with slope m through the point (x_0, y_0) is $y - y_0 = m(x - x_0)$.

1. Limits and Continuity

The exam expects you to know the definitions of the types of limits, and to be able to use these definitions.

- The definition of a limit: Let f be a function defined on both sides of a number a (though not necessarily at a), then f has limit L as x approaches a if $f(x)$ can be made as close as we like to L by choosing x close enough to a. This is written $\lim_{x \to a} f(x) = L$.
- The definition of one-sided limits, limits at infinity, infinite limits, *etc*, and the related notions of horizontal and vertical asymptotes.
- The different types of behaviour which limits can have: the left and right limits may be different, the a function can have a limit at a point without being defined at that point, *etc.*
- The limit laws: if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$ then:

$$\lim_{x \to a} (f + g)(x) = L + K, \quad \lim_{x \to a} cf(x) = cL,$$
$$\lim_{x \to a} (fg)(x) = LK, \quad \lim_{x \to a} (f(x))^m = L^m,$$
$$\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{K},$$

  where the last assumes $K \neq 0$, and the similar versions for limits at infinity.
- Special limits, such as $\lim_{x \to a} x^n = a^n$, $\lim_{x \to 0} \frac{\sin(x)}{x} = 0$, and $\lim_{x \to \infty} \frac{1}{x^n} = 0$ which are useful when finding the limit of a function.
- The techniques used to calculate limits: cancelling out common factors, dividing the numerator and denominator by a suitable power of x, and using the limits laws and known limits.
- Recognising when a limit might not exist, and how to show this is the case. For example, functions involving absolute values often have different left and right limits at certain points, and how to decide if an infinite limit is $+\infty$ or $-\infty$.
- The *squeeze theorem*: If f, g, h are functions and $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$. How to use the squeeze theorem to compute limits. An example of this type of question is Question 12 from the midterm.

You are also expected to understand continuous functions.

- A function f is *continuous* at a number a if $\lim_{x\to a} f(x) = f(a)$. The function is continuous on an interval I if it is continuous at every number in I.
- The similar definitions of one-sided continuity.
- The following functions are continuous at every number in their domain: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithm functions. You can use continuity of these functions when computing limits.
- If f and g are continuous functions then the functions $cf,\ f+g,\ fg,\ \frac{f}{g},\ f\circ g$ are continuous at every point in their domain (there are other similar statements saying when the composite function $f\circ g$ is continuous at a number a).
- You should know how to use left and right limits and continuity to deduce equations which must be satisfied for a function to be continuous, and be able to solve these equations. An example of this type of question is Question 9 from the midterm.
- The *intermediate value theorem*: Suppose f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and that N is between $f(a)$ and $f(b)$. Then there is $c \in (a, b)$ such that $f(c) = N$. How to use the intermediate value theorem to show that an equation has a real root. An example of this type of question is Question 11 from the midterm.

2. Derivatives

Most of our time has been spent on derivatives, and you should have a good understanding of what the derivative means, the techniques for differentiation, and the applications of differentiation.

- The definition of the derivative at a point: the derivative of a function f at a point a is the limit
$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
How the derivative can be viewed as a function, with domain all numbers a such that the above limit exists.
- The interpretations of the derivative as the rate of change in outputs of f compared to the inputs; the slope of the tangent line to the curve $f(x)$ at the point $(a, f(a))$; the velocity of a particle whose position at time t is given by $f(t)$.
- The different notation used for derivatives, including $\frac{d}{dx}f(x) = f'(x)$.
- The techniques for finding derivatives:
  - $\frac{d}{dx}(cf(x)) = cf'(x)$ and $\frac{d}{dx}(f+g)(x) = (f' + g'(x)$;
  - the *product rule*: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
  - the *quotient rule*: $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$;
  - the *chain rule*: $(f\circ g)'(x) = f'(g(x))g'(x)$;
  - *implicit differentiation*, which uses the chain rule;
  - *logarithmic differentiation*, which is a combination of implicit differentiation and the laws of logarithms;

  - how to combine the above rules to find the derivative of combinations of functions.
- The derivatives of the common functions:
  - power functions: $\frac{d}{dx}x^n = nx^{n-1}$ for n a positive whole number, or more generally $\frac{d}{dx}x^{\frac{p}{q}} = \frac{p}{q}x^{\frac{p}{q}-1}$;
  - exponential functions: $\frac{d}{dx}b^x = b^x \ln(b)$, in particular $\frac{d}{dx}e^x = e^x$;
  - trigonometric functions: $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}\tan(x) = \sec^2(x)$, and how to find the derivatives of the other trigonometric functions using the quotient rule;
  - logarithmic functions: $\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$, in particular $\frac{d}{dx}\ln(x) = \frac{1}{x}$;
  - inverse trigonometric functions: $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$ (it may be easier to learn how to derive these by implicit differentiation than learning the identities themselves, as they are easily confused) you will not be asked about the derivatives of the other inverse trigonometric functions, as they depend on the choice of domain for those functions.

We have also seen some ways derivatives can be used to solve problems.

- Related rates: if two quantities are related by an equation $B = f(x)$ (for example, B is an area or volume which depends on a length x, or B is a distance which depends on another distance x) then the derivative of this equation relates the rate of change of the two quantities.
- Approximation: the linear approximation to a function f at the point a is the function $L(x) = f(a) + f'(a)(x-a)$, which is useful because if x is close to a then the tangent to f at the point $(a, f(a))$, which is the line $L(x)$, is close to the curve f near a; equivalently, the differential $dy = f'(a)dx$, where dx is a small change from a to x, approximates the difference between $f(a)$ and $f(x)$ when $x = a + dx$, so $f(x)$ is approximately $f(a) + dy$.
- Curve sketching: Looking carefully at the formula of a function and its derivatives lets us deduce the shape of its graph:
  - x- and y-intercepts;
  - symmetry;
  - intervals of increase or decrease: f is increasing on the interval I if $f'(x) > 0$ for all $x \in I$, and f is decreasing on the interval I if $f'(x) < 0$ for all $x \in I$;
  - critical values: the critical values of f are the numbers c for which $f'(c) = 0$ or $f'(c)$ is undefined; Fermat's theorem says that if f has a local maximum or minimum at c and $f'(c)$ exists then $f'(c) = 0$;
  - asymptotes (see above);
  - inflection points are the points c with $f''(c) = 0$;
  - a graph is concave upward on an interval I if $f''(x) > 0$ for $x \in I$, which means the curve lies above its tangents on I, it is concave downward on I if $f''(x) < 0$ for $x \in I$, which means the curve lies below its tangents on I;
  - to sketch a graph mark all the above information clearly.

  Make sure you know how to decide if a function is positive or negative on an interval using the properties of sin and cos, x^2 and $(a - bx)(c + dx)$.

- Optimisation: if two quantities A and x are related by the equation $A = f(x)$ then the number c which gives a minimum or maximum value for A can be found by solving $\frac{dA}{dx} = 0$. Make sure you know how to check that a critical value gives a minimum or maximum using the first or second derivative test, or the closed interval method.
- Antidifferentiation: given a function f we can use our knowledge of the derivatives of common functions and the derivative rules to find a function F with $F' = f$. These rules show that the function f has many antiderivatives, the general form is $F(x) + c$. The antiderivative of an acceleration function is the velocity function, and the antiderivative of a velocity function is the position function. We can use extra information about the function F to deduce the constant c.
- The extreme value theorem: if f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum $f(c)$ and an absolute minimum $f(d)$ for some numbers c and d in $[a, b]$. The closed interval method allows us to find the absolute maximum and minimum of f on $[a, b]$ by checking for the largest and smallest of $f(a), f(b)$ and $f(c)$ for any critical values c.
- Rolle's theorem: suppose f is a continuous function on the closed interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$; then there is $c \in (a, b)$ with $f'(c) = 0$. Make sure you know how to use Rolle's theorem to show an equation has a unique root, or no more than two roots.
- Mean value theorem: let f be a continuous function on the closed interval $[a, b]$, differentiable on (a, b); Then there is a number $c \in (a, b)$ such that $f'(c)(b - a) = f(b) - f(a)$.

3. Word problems

The sections of the textbook on problem solving strategies offer lots of good tips for these questions.

You are expected to know the formulas for the perimeters, areas and volumes of common shapes for the word problems. Some of these formulas are below:

- perimeter of a shape: the sum of the lengths of the sides;
- area of a triangle: $\frac{1}{2}bh$, where b is the length of the base and h;
- perimeter of a circle: $2\pi r$, where r is the radius;
- area of a circle: πr^2, where r is the radius;
- volume of a sphere: $\frac{4}{3}\pi r^3$, where r is the radius;
- volume of a cone: $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the perpendicular height;
- volume of a triangular prism: $\frac{1}{2}bhl$, where b is the base of the triangular face, h its perpendicular height, and l its length;
- surface area of a cylinder: $2\pi rh + 2\pi r^2$, where r is the radius, h the height;
- volume of a cylinder: $\pi r^2 h$, where r is the radius and h the height.

For the optimisation and related rates problems it is often useful to draw a diagram and label everything carefully. Remember that optimisation problems are the ones which involve setting a derivative equal to 0; for these problems it is important that you know how to test if a critical value gives you a max or a min using the first or second derivative test, or the closed interval method.