MATH 110 LIMITS AND DIFFERENTIATION QUESTIONS

ANDREW MCKEE

This gives a few more examples of exam style questions and how they will be marked. Do NOT think the questions below are hints for what will be on the exam. The questions are random ones to give you an idea of how the mark scheme works, and how you should write your answers. This is also not an exhaustive list of the types of question that can appear on the exam, other types of question appear in the examples from the lectures, on the midterm, and in the past exams; make sure you know how to approach every type of question we have seen in the lectures.

Limits

There are many types of limit questions in the past papers uploaded and in the homework assignments. Here is a small sample of harder ones.

Question 1. *(3 marks)* $\lim_{x\to 2} \frac{\sqrt{2x}-x}{2-x}$

Solution. Factorise: (0,0.5,1) factorising correctly

$$\frac{\sqrt{2x}-x}{2-x} = \frac{\sqrt{x}(\sqrt{2}-\sqrt{x})}{\sqrt{2}^2-\sqrt{x}^2} = \frac{\sqrt{x}(\sqrt{2}-\sqrt{x})}{(\sqrt{2}-\sqrt{x})(\sqrt{2}+\sqrt{x})}$$

Since when taking the limit we never consider $x = 2$ we may cancel the $\sqrt{2}-\sqrt{x}$ terms as they are not zero. (0,0.5,1) cancelling Thus

$$\lim_{x\to 2} \frac{\sqrt{2x}-x}{2-x} = \lim_{x\to 2} \frac{\sqrt{x}(\sqrt{2}-\sqrt{x})}{(\sqrt{2}-\sqrt{x})(\sqrt{2}+\sqrt{x})} = \lim_{x\to 2} \frac{\sqrt{x}}{\sqrt{2}+\sqrt{x}}.$$

The limit laws tell us that the latter can be found by substituting, so (0,0.5,1) substitute to answer

$$\lim_{x\to 2} \frac{\sqrt{2x}-x}{2-x} = \lim_{x\to 2} \frac{\sqrt{x}}{\sqrt{2}+\sqrt{x}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}.$$

□

Question 2. *(3 marks)* $\lim_{x\to 0} \frac{\sin^2(5x)}{2x^2}$

Solution. We have

$$\frac{\sin^2(5x)}{2x^2} = \frac{1}{2}\left(\frac{\sin(5x)}{x}\right)^2 = \frac{1}{2}\left(\frac{5\sin(5x)}{5x}\right)^2 = \frac{25}{2}\left(\frac{\sin(5x)}{5x}\right)^2$$

Therefore, by the properties of limits, (0,0.5,1) rearranging

$$\lim_{x\to 0} \frac{\sin^2(5x)}{2x^2} = \frac{25}{2}\left(\lim_{x\to 0}\left(\frac{\sin(5x)}{5x}\right)\right)^2$$

We know that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$, so taking $\theta=5x$ (0,0.5,1) use of this limit

$$\lim_{x\to 0}\frac{\sin^2(5x)}{2x^2}=\frac{25}{2}\left(\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\right)\right)^2=\frac{25}{2}(1^2)=\frac{25}{2}.$$

(0,0.5,1) answer □

Derivatives

In each case find the derivative $\frac{dy}{dx}$. Make sure you know the chain, product and quotient rules as well as how to find the derivatives of polynomials, exponentials, logarithms, trig and inverse trig functions, etc. Most questions involve some combination of the chain, product and quotient rules as well as remembering the derivatives of the standard functions we have seen.

You do not need to say which rule you use for each derivative to get the marks, but it might help you keep track of what you are doing.

Question 3. *(3 marks)* $y=\sin^{-1}(x^2)$

Solution. This is a composition $y=f\circ g$ with $f(z)=\sin^{-1}(z)$ and $g(x)=x^2$. We have $g'(x)=2x$ and $f'(z)=\frac{1}{\sqrt{1-z^2}}$, so by the chain rule (0,0.5,1) intermediate derivatives (0,0.5,1) chain rule

$$y'=f'(g(x))g'(x)=\frac{1}{\sqrt{1-(x^2)^2}}2x=\frac{2x}{\sqrt{1-x^4}}$$

(0,0.5,1) answer
Note: if you cannot remember the derivative of the inverse sin function then at least remember the technique to work it out. We have $y=\sin^{-1}(x)$, so $\sin(y)=x$, and now we use implicit differentiation to find $\frac{dy}{dx}$: taking the derivative of both sides of $\sin(y)=x$, remembering y depends on x and using the chain rule, gives$\cos(y)\frac{dy}{dx}=1$. We know $\sin(y)=x$, so use the identity $\sin^2(y)+\cos^2(y)=1$ to write $\cos(y)=\sqrt{1-\sin^2(y)}$; since $\sin(y)=x$ then $\cos(y)=\sqrt{1-x^2}$. Hence

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

□

Question 4. *(3 marks)* $y=\log_{10}(x^2\sin(x))$

Proof. This is a composition $y=f\circ g$ with $f(z)=\log_{10}(z)$ and $g(x)=x^2\sin(x)$. We have $f'(z)=\frac{1}{z\ln(10)}$. To find $g'(x)$ we need the product rule: $g'(x)=x^2\cos(x)+2x\sin(x)$. (0,0.5,1) intermediate derivatives By the chain rule (0,0.5,1) using chain rule

$$y'=f'(g(x))g'(x)=\frac{1}{x^2\sin(x)\ln(10)}(x^2\cos(x)+2x\sin(x)).$$

(0,0.5,1) answer □

Section III word problems

It might be worth looking at the sections of the textbook on problem solving strategies for tips on how to approach these problems.

For the related rates and optimisation problems it is helpful to know the formulas for area and volume of common shapes: cones, spheres, cylinders, a solid with triangular cross section, etc, as well as remembering that the distance between objects travelling in perpendicular directions is the hypotenuse of a right angled triangle.

Question 5. *(5 marks) A water tank has the shape of an inverted cone with circular base of radius 2m and height 4m. If water is being pumped in to the tank at the rate of 2 cubic metres per minute find the rate at which the water level is rising when the depth of the water is 3m.*

Solution. If V is the volume of water in the tank and h the depth we need to relate dV/dt and dh/dt. We know $V = \frac{1}{3}\pi r^2 h$, where h is the radius of the cone at water level. By similar triangles $r/h = 2/4$, so $r = h/2$; therefore (0,0.5,1,1.5,2) equation for V with reasoning

$$V = \frac{\pi h^3}{12}.$$

Hence (0,0.5,1) derivative

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}.$$

Rearranging gives (0,0.5,1) solving for dh/dt

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}.$$

Substituting $h = 3$ and $dV/dt = 2$ we obtain $\frac{dh}{dt} = \frac{8}{9\pi}$. (0,0.5,1) substitute for answer □

Question 6. *Car A is travelling west at 50mph and car B is travelling north at 60mph. At what rate are they approaching each other when car A is 0.3 miles and car B is 0.4 miles from the intersection?*

Solution. Let x be the distance from A to the intersection and y the distance from B to the intersection. Then $dx/dt = -50$ and $dy/dt = -60$ (negative derivatives because distance decreasing). (0,0.5,1) correct information The distance between the cars is z, and the Pythagorean theorem gives $z^2 = x^2 + y^2$. (0,0.5,1) distance between cars Differentiating each side with respect to time gives (0,0.5,1)

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

so (0,0.5,1) solving for dz/dt

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

When $x = 0.3$ and $y = 0.4$ the relation between x, y, z above gives $z = 0.5$. (0,0.5,1) finding z Substituting we find (0,0.5,1) substitute to answer

$$\frac{dz}{dt} = 2(0.3(-50) + 0.4(-60)) = -78,$$

so they are approaching each other at 78mph. □

Question 7. *(7 marks) A piece of wire 10m long is cut in two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so the total area of the square and the triangle is a maximum?*

Solution. Let x be the length of wire used for the square. The total area is (0,0.5,1,1.5,2) finding area correctly

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right) = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(10-x)^2$$

since the triangle has side length $(10-x)/3$ and perpendicular height $\sqrt{3}(10-x)/6$. Hence (0,0.5,1) derivative

$$A'(x) = \frac{x}{8} - \frac{\sqrt{3}}{18}(10-x).$$

The only critical value is therefore $x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$. (0,0.5,1) critical value To check that this is a minimum we note that the formula $A(x)$ is defined on the interval $[0,10]$ since this is the range for x, so we use the closed interval test: $A(0) = (\sqrt{3}/36)100$, $A(10) = 100/16$ and $A(\frac{40\sqrt{3}}{9+4\sqrt{3}}) \approx 2.72$. (0,0.5,1) suitable test (0,0.5,1) is a min Hence the minimum occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$. (0,0.5,1) answer □

Question 8. *(7 marks) A boat leaves a dock at 2pm and travels due south at 20kmph. Another boat has been heading due east at 15kmph and reaches the same dock at 3pm. At what time were the boats closest together?*

Solution. Let t be the time, in hours, after 2pm. We assume the dock is the origin $(0,0)$. The boat heading south at time t has position $(0,-20t)$; the boat heading east has position at time t given by $(-15+15t,0)$. (0,0.5,1) setup Let $D(t)$ be the distance between the boats at time t; by the Pythagorean theorem (or distance formula) the distance D between the boats at time t is (0,0.5,1) distance formula

$$D(t)^2 = 20^2t^2 + 15^2(t-1)^2.$$

Since distance is positive minimising $D(t)$ is the same as minimising $f(t) = D(t)^2$. We have (0,0.5,1) derivative of appropriate function

$$f'(t) = 800t + 450(t-1),$$

so the critical value occurs when $0 = f'(t)$, equivalently $t = 450/1250 = 9/25$. (0,0.5,1,1.5) finding the critical value Since $f''(t) = 800+450 > 0$ this critical value is a local minimum by the second derivative test; moreover, we observe $f'(t) < 0$ when $t < 9/25$ and $f'(t) > 0$ when $t > 9/25$, so this local minimum is an absolute minimum. (0,0.5,1,1.5) checking for min So the boats are closest together at $t = 9/25$ hours after 2pm, which is 21min 36sec after 2pm. Therefore the boats are closest together at 2:21:36pm. (0,0.5,1) final answer □

Remember that for a curve $y = f(x)$ the derivative $\frac{dy}{dx} = y'$ evaluated at a represents the slope of the tangent to the curve at the point $(a, f(a))$. Some questions will test your understanding of tangent lines.

Question 9. *(5 marks) Use implicit differentiation to find an equation of the tangent line to the curve $x^2 - xy - y^2 = 1$ at the point $(2,1)$.*

Proof. Implicit differentiation gives (0,0.5,1,1.5,2) implicit differentiation

$$2x - xy' - y - 2yy' = 0.$$

therefore $y'(x+2y) = 2x - y$, so $y' = \frac{2x-y}{x+2y}$. (0,0.5,1) correct y'

The slope of the tangent to the curve at the point $(2,1)$ is found by plugging in $x = 2,\ y = 1$ to the expression for y': slope is $(4-1)/(2+2) = 3/4$. (0,0.5,1) finding slope correctly Using the general equation for a line with known slope m and a known point (x_1, y_1): $y - y_1 = m(x - x_1)$ we have $y - 1 = \frac{3}{4}(x-2)$, or $y = \frac{3}{4}x - \frac{1}{2}$. (0,0.5,1) finding line equation correctly □

You are expected to know the formulas involved in linear approximation or differentials, and be able to apply them.

Question 10. *(5 marks) Use linear approximation or differentials to estimate* $\tan(46°)$*.*

Solution. Solution by linear approximation: We know that $45° = \pi/4$ radians, and $\tan(\pi/4) = 1$, so choose $f(x) = \tan(x)$ and $a = 45°$. (0,0.5,1) appropriate choices of f and a Since x is measured in degrees we need the chain rule to find the derivative of $\tan(x°)$: (0,0.5,1,1.5) derivative

$$f'(x) = \frac{d}{dx}(\tan(x°)) = \frac{d}{dx}(\tan(\frac{\pi}{180}x)) = \frac{\pi}{180}\sec^2(\frac{\pi}{180}x) = \frac{\pi}{180}\sec^2(x°).$$

The linearisation of f at a is (0,0.5,1,1.5) linearisation correct

$$L(x) = f(a) + f'(a)(x-a) = 1 + \frac{\pi}{180}\sec^2(45°)(x-45) = 1 + \frac{\pi}{180}2(x-45)$$

Hence $\tan(46°)$ is approximately $L(46) = 1 + \frac{\pi}{180}2(46-45) = 1 + \frac{\pi}{90}$ (0,0.5,1) answer (since π is approximately 3 this is about $1 + 1/30$).

Solution by differentials: We know that $45° = \pi/4$ radians, and $\tan(\pi/4) = 1$, so choose $f(x) = \tan(x)$ and $a = 45°$, so $dx = 1°$. (0,0.5,1,1.5) appropriate choices As above we find $f'(x)$. (0,0.5,1,1.5) marking Hence (0,0.5,1) correct dy

$$dy = f'(a)dx = \frac{2\pi}{180}.$$

It follows that $\tan(45°)$ is approximately $f(a) + dy = 1 + \frac{\pi}{90}$. (0,0.5,1) answer □

For antiderivative questions make sure you know how to use your knowledge of the derivatives of common functions to find the antiderivatives of common functions.

Question 11. *(6 marks) Find the function f if $f''(x) = x^2 - \cos(2x)$, $f'(0) = 1$ and $f(0) = 3$.*

Solution. To find f', first note that $\frac{d}{dx}(x^3/3) = x^2$. To find a function that has derivative $-\cos(2x)$ we remember that the derivative of $\sin(2x)$ is $2\cos(2x)$ (using the chain rule), so $-\cos(2x)$ is the derivative of $-\frac{1}{2}\sin(2x)$. (0,0.5,1,1.5) reasoning for f' The general form of f' is therefore

$$f'(x) = \frac{x^3}{3} - \frac{1}{2}\sin(2x) + c.$$

To find c we have $f'(0) = c$ and we are given $f'(0) = 1$, so $c = 1$. (0,0.5,1) finding constant Thus (0,0.5) final form of f'

$$f'(x) = \frac{x^3}{3} - \frac{1}{2}\sin(2x) + 1.$$

To find f, first note that $\frac{d}{dx}(x^4/12) = x^3/3$. To find a function that has derivative $-\sin(2x)/2$ we remember that the derivative of $\cos(2x)$ is $-2\sin(2x)$ (using the chain rule), so $-\sin(2x)/2$ is the derivative of $\frac{1}{4}\cos(2x)$, and 1 is the derivative of x.(0,0.5,1,1.5) reasoning for f The general form of f is therefore

$$f(x) = \frac{x^4}{12} + \frac{1}{4}\cos(2x) + x + d.$$

To find d we have $f(0) = 1/4$ (since $\cos(0) = 1$) and we are given $f(0) = 3$, so $d = 11/4$. (0,0.5) finding constant Thus (0,0.5,1) final form of f

$$f(x) = \frac{x^4}{12} + \frac{1}{4}\cos(2x) + x + \frac{11}{4}.$$

□

Question 12. *(6 marks) A particle is moving with acceleration $a(t) = t^2 - 4t + 6$. Find the position function $s(t)$ if $s(0) = 0$ and $s(1) = 20$.*

Solution. We know that $a(t)$ is the second derivative $s''(t)$ of $s(t)$, so we start by finding $v(t) = s'(t)$. By the power rule (0,0.5,1,1.5) finding s'

$$s'(t) = \frac{t^3}{3} - 2t^2 + 6t + c.$$

We do not have any data that can be used to find c, so we use the power rule again to find $s(t)$: (0,0.5,1,1.5) finding s

$$s(t) = \frac{t^4}{12} - \frac{2t^3}{3} + 3t^2 + ct + d.$$

Now we find c and d: $s(0) = d$ and we are told $s(0) = 0$, so $d = 0$. (0,0.5,1) find d Since $s(1) = 1/12 - 2/3 + 3 + c$ and we are told $s(1) = 1$ we have $c = -17/12$. (0,0.5,1) find c Thus (0,0.5,1) final answer

$$s(t) = \frac{t^4}{12} - \frac{2t^3}{3} + 3t^2 - \frac{17t}{12}.$$

□

Curve sketching questions expect you to know how to find all the features of a curve we have discussed in the lectures: local maxima and minima and critical numbers, horizontal and vertical asymptotes, points of inflection and concavity. You are also expected to use these to sketch the curve.

Question 13. *Let $f(x) = (4 - x^2)^5$.*

(a) (3 marks) Compute f' and f''.
(b) (2 marks) Find all local maxima and minima.
(c) (2 marks) Determine the intervals where f is increasing or decreasing.
(d) (2 marks) Find all inflection points.
(e) (2 marks) Determine the intervals where f is concave upward or downward.
(f) (1 mark) Find all horizontal and vertical asymptotes of f.
(g) (3 marks) Sketch the graph of f indicating all the above information.

Solution. Using the chain and quotient rules we find $f'(x) = -10x(4 - x^2)^4$ and $f''(x) = -10(4 - x^2)^3(4 - 9x^2)$. (0,0.5,1) f' (0,0.5,1,1.5,2) f'' using product and chain rule

Maxima/minima: $f'(x) = 0$ when $x = 0$, $x = 2$ or $x = -2$. (0,0.5,1) solve $f'(x) = 0$ The second derivative test shows that $x = 0$ is a maximum, the others are not. Local maximum at $f(0) = 1024$. (0,0.5,1) correctly stating only max

Increasing/decreasing: Since $f'(x) > 0$ if and only if $x < 0$ we have x is increasing on $(-\infty, 0)$ (0,0.5,1) increasing with reason and $f'(x) < 0$ if and only if $x > 0$ we have f is decreasing on $(0, \infty)$. (0,0.5,1) decreasing with reason

Inflection points: We can see $f''(x) = 0$ when $x = 2, -2, \frac{2}{3}, -\frac{2}{3}$. (0,0.5,1) solving $f''(x) = 0$ So the inflection points are $(2, 0)$, $(-2, 0)$, $(\frac{2}{3}, f(2/3))$, $(-2/3, f(-2/3))$. (0,0.5,1) correct points

Concavity: Inspecting f'' we see $f''(x) > 0$ if and only if $-2 < x < -2/3$ of $2/3 < x < 2$ and $f''(x) < 0$ if and only if $x < -2$ or $x > 2$ or $-2/3 < x < 2/3$. (0,0.5,1) Thus f is concave upwards on $(-\infty, 2)$ and $(-2/3, 2/3)$ and $(2, \infty)$ and concave downwards on $(-2, -2/3)$ and $(2, 2/3)$. (0,0.5,1)

Asymptotes: No asymptotes. (0,0.5,1)

Sketch: (0.5) general shape (0.5) each point above □