## Approximation properties for group actions via multipliers

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IPM Tehran January 2020 Algebras associated to a group

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- ▶ group von Neumann algebra vN(G) =  $C_r^*(G)'' = \{\lambda_s : s \in G\}'';$
- Fourier algebra A(G): Banach algebra (pointwise operations) of coeffecients of λ

$$v: G \to \mathbb{C}; \ v(s) = \langle \lambda_s \xi, \eta \rangle \ \text{for some } \xi, \eta \in \ell^2(G).$$

We have  $A(G)^* = vN(G)$ .

It is often useful for us to think of operators in  $\mathcal{B}(\ell^2(G))$  as matrices over  $\mathbb{C}$  indexed by  $G \times G$ :

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The matrix of  $x \in C_r^*(G)$  is constant down the diagonals. This means that  $x_{s,t}$  depends only on  $st^{-1}$ .

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & x_{e} & x_{r} & \ddots & \\ & x_{r-1} & x_{e} & x_{r} & \\ & \ddots & x_{r-1} & x_{e} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The functional  $x \mapsto \langle x \delta_e, \delta_e \rangle = x_e$  is a tracial state on  $C_r^*(G)$ , acting by

$$\sum_{r\in G}a_r\lambda_r\mapsto a_e.$$

Theorem (Lance '74)

A discrete group G is amenable if and only if  $C_r^*(G)$  is nuclear.

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Nuclearity: there exists a net of finite rank, completely positive contractions  $(\Phi_i : A \to A)_i$  with  $||\Phi_i(a) - a|| \to 0$   $(a \in A)$ . Amenability: there exists a net of finitely supported, normalised positive-definite functions  $(u_i : G \to \mathbb{C})_i$  with  $u_i \to 1$  pointwise.

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#### Proof.

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A Herz–Schur multiplier is a function  $u : G \to \mathbb{C}$  such that

$$S_u: C_r^*(G) \to C_r^*(G); \ S_u\left(\sum_{r \in G} a_r \lambda_r\right) = \sum_{r \in G} u(r) a_r \lambda_r$$

is completely bounded. Characterisation:  $u(st^{-1}) = \langle V(s), W(t) \rangle$ .

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Let  $(u_i : G \to \mathbb{C})_i$  be the finitely supported, normalised positive-definite functions approximating the constant function 1, from amenability of *G*. Viewing  $(u_i)_i$  as **Herz–Schur multipliers**:

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 $u_i$  finite support  $\implies S_{u_i}$  finite rank  $u_i$  normalised pos.-def. Herz–Schur  $\implies S_{u_i}$  contractive comp. pos.  $u_i \rightarrow 1$  pointwise  $\implies S_{u_i} \rightarrow \mathrm{id}$  point-norm

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Conversely, if  $(\Phi_i)_i$  implement nuclearity of  $C_r^*(G)$  define

$$u_i: G \to \mathbb{C}; \ u_i(r) := \langle \Phi_i(\lambda_r) \lambda_r^* \delta_e, \delta_e \rangle.$$

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Haagerup's idea: adjust the conditions on the  $u_i$  and see what properties of  $C_r^*(G)$  the  $S_{u_i}$  implement. Examples:

- weak amenability forget about positivity condition (just keep uniform boundedness), gives CBAP of C<sup>\*</sup><sub>r</sub>(G);
- Haagerup property require  $C_0(G)$  positive-definite functions.

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- ► free groups are the best-known non-amenable groups, but F<sub>2</sub> is not too bad it is weakly amenable and has Haagerup property.
- C<sup>\*</sup><sub>r</sub>(𝔽<sub>2</sub>) is not nuclear (Lance), but as 𝔽<sub>2</sub> is weakly amenable this C<sup>\*</sup>-algebra has the CBAP, and therefore Grothendieck's MAP, answering an open question.

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- ▶ Open problem: G weakly amenable with Λ<sub>WA</sub>(G) = 1 ⇒ G Haagerup property. (The converse is known to be false.)

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From  $(A, G, \alpha)$  we form **reduced crossed product**  $A \rtimes_{\alpha, r} G$ : finite sums  $\sum_{r \in G} a_r \lambda_r$ ,  $a_r \in A$ , closed in operator norm of  $\mathcal{B}(\ell^2(G) \otimes \mathcal{H})$   $(A \subseteq \mathcal{B}(\mathcal{H}))$ .

Action encoded by  $\lambda_s a \lambda_s^* = \alpha_s(a)$ . When  $A = \mathbb{C}$  this is  $C_r^*(G)$ .

## $\label{eq:matrix} \begin{array}{l} \mbox{Matrix viewpoint} \\ \mathcal{T} \in \mathcal{B}(\ell^2(\mathcal{G}) \otimes \mathcal{H}) \mbox{ also has a matrix representation} \end{array}$

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The matrix of  $x \in A \rtimes_{\alpha,r} G$  has a **diagonal pattern**.

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & \alpha_{s}(x_{e}) & \alpha_{s}(x_{r}) & \ddots & \\ & x_{r-1} & x_{e} & x_{r} & \\ & \ddots & \alpha_{s-1}(x_{r-1}) & \alpha_{s-1}(x_{e}) & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}$$

The map  $x \mapsto x_e$  is a conditional expectation  $\mathcal{E} : A \rtimes_{\alpha, r} G \to A$ , acting by

$$\mathcal{E}: \sum_{r\in \mathcal{G}} a_r \lambda_r \mapsto a_e.$$

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**Option 2** (MSTT): develop Herz–Schur multipliers of dynamical systems, then follow the Haagerup programme.

#### Multipliers of crossed products

With Todorov and Turowska we developed Herz–Schur multipliers of a dynamical system.

Recall:  $u: G \to \mathbb{C}$  is a Herz–Schur multiplier if map is completely bounded

$$S_u: C_r^*(G) \to C_r^*(G); \ S_u\left(\sum_{r \in G} a_r \lambda_r\right) = \sum_{r \in G} u(r)a_r \lambda_r, \quad a_r \in \mathbb{C}.$$

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#### Definition

 $F: G \to CB(A)$  is a Herz–Schur multiplier of  $(A, G, \alpha)$  if map is completely bounded

$$S_F: A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G; \ S_F\left(\sum_{r \in G} a_r \lambda_r\right) = \sum_{r \in G} F(r)(a_r)\lambda_r, \quad a_r \in A.$$

Characterisation: F is a Herz–Schur  $(A, G, \alpha)$ -multiplier if and only if  $\alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a))) = W(s)^*\rho(a)V(t)$ .  $S_F$  completely positive if and only if V = W. Haagerup programme for dynamical systems

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So far we (M., Skalski, Todorov, Turowska) have written down the correct conditions for:

- weak amenability of  $(A, G, \alpha)$ , *i.e.* CBAP of  $A \rtimes_{\alpha, r} G$ ;
- nuclearity of  $(A, G, \alpha)$ , *i.e.* nuclearity of  $A \rtimes_{\alpha, r} G$ ;
- Haagerup property of (A, G, α), *i.e.* C<sup>\*</sup> Haagerup property of A ⋊<sub>α,r</sub> G;
- AP of  $(A, G, \alpha)$ , *i.e.* SOAP of  $A \rtimes_{\alpha, r} G$ .

Theorem (MSTT '18)

 $(A, G, \alpha)$  a C\*-dynamical system, G discrete. TFAE:

i. there are Herz–Schur  $(A, G, \alpha)$ -multipliers  $(F_i)_i$  satisfying:

a.  $F_i$  positive and  $||F_i(e)||_{cb} \le 1$ ; b.  $F_i(r)$  finite rank, non-zero for only finitely many  $r \in G$ ; c.  $||F_i(r)(a) - a|| \to 0$  for all  $r \in G, a \in A$ .

ii.  $A \rtimes_{\alpha,r} G$  is nuclear.

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Proof.

(i)  $\Longrightarrow$  (ii) As in Lance's proof  $(S_{F_i})_i$  implement nuclearity:

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(ii)  $\Longrightarrow$  (i) If  $(\Phi_i)_i$  implement nuclearity of  $A \rtimes_{\alpha,r} G$  then define  $F_i(r)(a) := \mathcal{E}(\Phi_i(a\lambda_r)\lambda_r^*).$ 

Here is an example of option 1 above.

Theorem (Anantharaman-Delaroche, Brown–Ozawa)

 $(A, G, \alpha)$  a C\*-dynamical system, such that  $\alpha$  is an amenable action. A unital. If A is nuclear then  $A \rtimes_{\alpha,r} G$  is nuclear.

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## Proof (MSTT).

*G* acts amenably means there are  $(T_i : G \to Z(A)^+)_i$  finitely supported,  $\sum_{r \in G} T_i(r)^2 = 1_A$ , and for each  $t \in G$ 

$$\left\|\sum_{r\in G} (T_i(r) - \alpha_t(T_i(t^{-1}r)))^*(T_i(r) - \alpha_t(T_i(t^{-1}r)))\right\| \to 0.$$

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Let  $(\Phi_j : A \to A)_j$  implement nuclearity of A. Define

$$F_{i,j}(r)(a) := \sum_{p \in G} T_i(p) \alpha_p(\Phi_j(\alpha_{p^{-1}}(a))) \alpha_r(T_i(r^{-1}p)),$$

Herz–Schur  $(A, G, \alpha)$  multipliers satisfying the above Theorem.

## Schur multipliers

Definition (MTT)  $\varphi: G \times G \rightarrow CB(A)$  is a Schur A-multiplier if the following map is completely bounded.

 $S_{\varphi}: \mathcal{K}(\ell^{2}(G)) \otimes A \rightarrow \mathcal{K}(\ell^{2}(G)) \otimes A; \ (S_{\varphi}(T))_{s,t} := \varphi(s,t)(T_{s,t})$ 

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### Theorem (MTT)

- $\varphi: G \times G \rightarrow C\mathcal{B}(A)$ . The following are equivalent:
  - i.  $\varphi$  is a Schur A-multiplier;
  - ii.  $\varphi(s,t)(a) = W(s)^* \rho(a) V(t)$ , where  $\rho : A \to \mathcal{B}(\mathcal{H}_{\rho})$  is a representation and  $V, W : G \to \mathcal{B}(\mathcal{H}, \mathcal{H}_{\rho})$  are bounded.

Moreover,  $S_{\varphi}$  is completely positive if and only if V = W.

## Schur multipliers

Definition (MTT)  $\varphi: G \times G \rightarrow CB(A)$  is a Schur A-multiplier if the following map is completely bounded.

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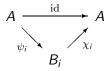
ii.  $\varphi(s,t)(a) = W(s)^* \rho(a) V(t)$ , where  $\rho : A \to \mathcal{B}(\mathcal{H}_{\rho})$  is a representation and  $V, W : G \to \mathcal{B}(\mathcal{H}, \mathcal{H}_{\rho})$  are bounded.

Moreover,  $S_{\varphi}$  is completely positive if and only if V = W.

Note: *F* is a Herz–Schur (*A*, *G*,  $\alpha$ )-multiplier if and only if  $\mathcal{N}(F)(s, t)(a) := \alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a)))$  is a Schur *A*-multiplier.

#### Exactness

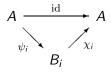
Nuclearity can also be viewed as existence of an **approximate** factorisation of the identity map  $id : A \rightarrow A$ :



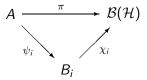
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#### Exactness

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 $B_i$  finite-dimensional,  $\psi_i, \chi_i$  contractive comp. pos.,  $\chi_i \circ \psi_i \rightarrow \text{id.}$ A is exact if there is a faithful representation  $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$  which has an **approximate factorisation**:



 $B_i$  finite-dimensional,  $\psi_i, \chi_i$  contractive comp. pos.,  $\chi_i \circ \psi_i \to id$ .

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We can use Schur multipliers to characterise exactness of  $A \rtimes_{\alpha,r} G$ . Theorem (MT)

 $(A, G, \alpha)$  a C\*-dynamical system, G discrete. TFAE:

i. there are (positive) Schur A-multipliers  $(\varphi_i)_i$  such that:

# a. sup<sub>i</sub> ||φ<sub>i</sub>||<sub>☉</sub> ≤ ∞; b. φ<sub>i</sub> supported on {(s,t) : st<sup>-1</sup> ∈ K<sub>i</sub> a finite set}; c. ||φ<sub>i</sub>(s,t)(α<sub>s-1</sub>(a)) - α<sub>s-1</sub>(a)|| → 0 uniformly on a strip (a ∈ A); d. the space {φ<sub>i</sub>(s,r<sup>-1</sup>s)(α<sub>s-1</sub>(a)) : a ∈ A, s,t ∈ G} is finite-dimensional;

ii.  $A \rtimes_{\alpha,r} G$  is exact.

The  $(S_{\varphi_i})_i$  give external approximations of  $A \rtimes_{\alpha,r} G$ : they do not preserve the **diagonal pattern**.

## Exactness and nuclearity

We can prove Ozawa's result on exactness of discrete groups.

Theorem (Ozawa '00)

G discrete group. The following are equivalent:

- i.  $C_r^*(G)$  is exact;
- ii. the uniform Roe algebra  $\ell^{\infty}(G) \rtimes_{\beta,r} G$  is nuclear.

## Exactness and nuclearity

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- ii. the uniform Roe algebra  $\ell^{\infty}(G) \rtimes_{\beta,r} G$  is nuclear.

#### Proof (MT).

(ii)  $\Longrightarrow$  (i) is trivial because  $C_r^*(G) \subset \ell^{\infty}(G) \rtimes_{\beta,r} G$ . (i)  $\Longrightarrow$  (ii) Take positive Schur multipliers  $(\varphi_i : G \times G \to \mathbb{C})_i$  giving exactness. Identify with

$$\phi_i: G \to \ell^\infty(G); \ \phi_i(r)(s) := \varphi_i(s, r^{-1}s).$$

These are Herz–Schur ( $\ell^{\infty}(G), G, \beta$ )-multipliers which satisfy our nuclearity conditions.

Thank you for listening!