Approximation properties for group actions via multipliers

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Algebras associated to a group

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The **left regular representation** of G is

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- group von Neumann algebra
  $\mathrm{vN}(G) = C_r^*(G)'' = \{\lambda_s : s \in G\}''$;
- Fourier algebra $A(G)$: Banach algebra (pointwise operations) of coeffecients of λ

$$v : G \to \mathbb{C}; \ v(s) = \langle \lambda_s \xi, \eta \rangle \ \text{for some } \xi, \eta \in \ell^2(G).$$

We have $A(G)^* = \mathrm{vN}(G)$.

It is often useful for us to think of operators in $\mathcal{B}(\ell^2(G))$ as matrices over $\mathbb{C}$ indexed by $G \times G$:

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The matrix of $x \in C_r^*(G)$ is **constant down the diagonals**. This means that $x_{s,t}$ depends only on st^{-1}.

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & x_e & x_r & \ddots & \\ & x_{r^{-1}} & x_e & x_r & \\ & \ddots & x_{r^{-1}} & x_e & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The functional $x \mapsto \langle x\delta_e, \delta_e \rangle = x_e$ is a tracial state on $C_r^*(G)$, acting by

$$\sum_{r \in G} a_r \lambda_r \mapsto a_e.$$

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Nuclearity: there exists a net of finite rank, completely positive contractions $(\Phi_i : A \to A)_i$ with $\|\Phi_i(a) - a\| \to 0$ $(a \in A)$.

Amenability: there exists a net of finitely supported, normalised positive-definite functions $(u_i : G \to \mathbb{C})_i$ with $u_i \to 1$ pointwise.

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Proof.

Amenable groups have a net of finitely supported, positive-definite, Herz–Schur multipliers approximating the constant function 1. $\square$

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Herz–Schur multipliers

A Herz–Schur multiplier is a function $u : G \to \mathbb{C}$ such that

$$S_u : C_r^*(G) \to C_r^*(G);\ S_u\left(\sum_{r \in G} a_r \lambda_r\right) = \sum_{r \in G} u(r) a_r \lambda_r$$

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Let $(u_i : G \to \mathbb{C})_i$ be the finitely supported, normalised positive-definite functions approximating the constant function 1, from amenability of G. Viewing $(u_i)_i$ as **Herz–Schur multipliers**:

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$$\begin{aligned} u_i \text{ finite support} &\implies S_{u_i} \text{ finite rank} \\ u_i \text{ normalised pos.-def. Herz–Schur} &\implies S_{u_i} \text{ contractive comp. pos.} \\ u_i \to 1 \text{ pointwise} &\implies S_{u_i} \to \mathrm{id} \text{ point-norm} \end{aligned}$$

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Conversely, if $(\Phi_i)_i$ implement nuclearity of $C_r^*(G)$ define

$$u_i : G \to \mathbb{C};\ u_i(r) := \langle \Phi_i(\lambda_r)\lambda_r^*\delta_e, \delta_e\rangle\,. \qquad \square$$

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Examples:

- weak amenability forget about positivity condition (just keep uniform boundedness), gives CBAP of $C_r^*(G)$;
- Haagerup property require $C_0(G)$ positive-definite functions.

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- Open problem: G weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 1 \stackrel{?}{\Longrightarrow} G$ Haagerup property. (The converse is known to be false.)

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From (A, G, α) we form **reduced crossed product** $A \rtimes_{\alpha,r} G$: finite sums $\sum_{r \in G} a_r \lambda_r$, $a_r \in A$, closed in operator norm of $\mathcal{B}(\ell^2(G) \otimes \mathcal{H})$ $(A \subseteq \mathcal{B}(\mathcal{H}))$.

Action encoded by $\lambda_s a \lambda_s^* = \alpha_s(a)$. When $A = \mathbb{C}$ this is $C_r^*(G)$.

Matrix viewpoint

$T \in \mathcal{B}(\ell^2(G) \otimes \mathcal{H})$ also has a matrix representation

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The matrix of $x \in A \rtimes_{\alpha,r} G$ has a **diagonal pattern**.

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \alpha_s(x_e) & \alpha_s(x_r) & \ddots & \\ & x_{r^{-1}} & x_e & x_r & \\ & \ddots & \alpha_{s^{-1}}(x_{r^{-1}}) & \alpha_{s^{-1}}(x_e) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The map $x \mapsto x_e$ is a conditional expectation $\mathcal{E} : A \rtimes_{\alpha,r} G \to A$, acting by

$$\mathcal{E} : \sum_{r \in G} a_r \lambda_r \mapsto a_e.$$

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Question: how are the approximation properties of $A \rtimes_{\alpha,r} G$ related to the properties of A and G?

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Option 1 (Brown–Ozawa): develop amenable actions, show that if G acts amenably then $A \rtimes_{\alpha,r} G$ has approximation property when A does.

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Option 2 (MSTT): develop Herz–Schur multipliers of dynamical systems, then follow the Haagerup programme.

Multipliers of crossed products

With Todorov and Turowska we developed **Herz–Schur multipliers of a dynamical system**.

Recall: $u : G \to \mathbb{C}$ is a Herz–Schur multiplier if map is completely bounded

$$S_u : C_r^*(G) \to C_r^*(G);\ S_u\left(\sum_{r\in G} a_r\lambda_r\right) = \sum_{r\in G} u(r)a_r\lambda_r, \quad a_r \in \mathbb{C}.$$

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Definition

$F : G \to \mathcal{CB}(A)$ is a Herz–Schur multiplier of (A, G, α) if map is completely bounded

$$S_F : A\rtimes_{\alpha,r}G \to A\rtimes_{\alpha,r}G;\ S_F\left(\sum_{r\in G} a_r\lambda_r\right) = \sum_{r\in G} F(r)(a_r)\lambda_r, \quad a_r \in A.$$

Characterisation: F is a Herz–Schur (A, G, α)-multiplier if and only if $\alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a))) = W(s)^*\rho(a)V(t)$. S_F completely positive if and only if $V = W$.

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We can now do the same for C^*-dynamical systems:

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So far we (M., Skalski, Todorov, Turowska) have written down the correct conditions for:

- weak amenability of (A, G, α), *i.e.* CBAP of $A \rtimes_{\alpha,r} G$;
- nuclearity of (A, G, α), *i.e.* nuclearity of $A \rtimes_{\alpha,r} G$;
- Haagerup property of (A, G, α), *i.e.* C^* Haagerup property of $A \rtimes_{\alpha,r} G$;
- AP of (A, G, α), *i.e.* SOAP of $A \rtimes_{\alpha,r} G$.

Theorem (MSTT '18)

(A, G, α) a C^*-dynamical system, G discrete. TFAE:

i. *there are Herz–Schur (A, G, α)-multipliers $(F_i)_i$ satisfying:*
   a. *F_i positive and $\|F_i(e)\|_{\mathrm{cb}} \leq 1$;*
   b. *$F_i(r)$ finite rank, non-zero for only finitely many $r \in G$;*
   c. *$\|F_i(r)(a) - a\| \to 0$ for all $r \in G, a \in A$.*

ii. *$A \rtimes_{\alpha,r} G$ is nuclear.*

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Proof.

(i) $\implies$ (ii) As in Lance's proof $(S_{F_i})_i$ implement nuclearity:

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(ii) $\implies$ (i) If $(\Phi_i)_i$ implement nuclearity of $A \rtimes_{\alpha,r} G$ then define

$$F_i(r)(a) := \mathcal{E}\big(\Phi_i(a\lambda_r)\lambda_r^*\big).$$

□

Amenable actions and nuclearity

Here is an example of **option 1** above.

Theorem (Anantharaman-Delaroche, Brown–Ozawa)

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Proof (MSTT).

G acts amenably means there are $(T_i : G \to Z(A)^+)_i$ finitely supported, $\sum_{r\in G} T_i(r)^2 = 1_A$, and for each $t \in G$

$$\left\| \sum_{r\in G} (T_i(r) - \alpha_t(T_i(t^{-1}r)))^*(T_i(r) - \alpha_t(T_i(t^{-1}r))) \right\| \to 0.$$

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Let $(\Phi_j : A \to A)_j$ implement nuclearity of A. Define

$$F_{i,j}(r)(a) := \sum_{p\in G} T_i(p)\alpha_p(\Phi_j(\alpha_{p^{-1}}(a)))\alpha_r(T_i(r^{-1}p)),$$

Herz–Schur (A, G, α) multipliers satisfying the above Theorem. □

Schur multipliers

Definition (MTT)

$\varphi : G \times G \to \mathcal{CB}(A)$ is a Schur A-multiplier if the following map is completely bounded.

$$S_\varphi : \mathcal{K}(\ell^2(G)) \otimes A \to \mathcal{K}(\ell^2(G)) \otimes A; \; (S_\varphi(T))_{s,t} := \varphi(s,t)(T_{s,t})$$

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Theorem (MTT)

$\varphi : G \times G \to \mathcal{CB}(A)$. *The following are equivalent:*

i. φ *is a Schur A-multiplier;*
ii. $\varphi(s,t)(a) = W(s)^*\rho(a)V(t)$, *where* $\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ *is a representation and* $V, W : G \to \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ *are bounded.*

Moreover, S_φ *is completely positive if and only if* $V = W$.

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Definition (MTT)

$\varphi : G \times G \to \mathcal{CB}(A)$ is a Schur A-multiplier if the following map is completely bounded.

$$S_\varphi : \mathcal{K}(\ell^2(G)) \otimes A \to \mathcal{K}(\ell^2(G)) \otimes A;\ (S_\varphi(T))_{s,t} := \varphi(s,t)(T_{s,t})$$

Theorem (MTT)

$\varphi : G \times G \to \mathcal{CB}(A)$. The following are equivalent:

i. *φ is a Schur A-multiplier;*
ii. *$\varphi(s,t)(a) = W(s)^*\rho(a)V(t)$, where $\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation and $V, W : G \to \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ are bounded.*

Moreover, S_φ is completely positive if and only if $V = W$.

Note: F is a **Herz–Schur** (A, G, α)-**multiplier** if and only if $\mathcal{N}(F)(s,t)(a) := \alpha_{s^{-1}}(F(st^{-1})(\alpha_s(a)))$ is a **Schur** A-**multiplier**.

Exactness

Nuclearity can also be viewed as existence of an **approximate factorisation** of the identity map $\mathrm{id} : A \to A$:

$$\begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "\psi_i"'] & & A \\ & B_i \arrow[ur, "\chi_i"'] & \end{tikzcd}$$

B_i finite-dimensional, ψ_i, χ_i contractive comp. pos., $\chi_i \circ \psi_i \to \mathrm{id}$.

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A is exact if there is a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$ which has an **approximate factorisation**:

$$\begin{tikzcd} A \arrow[rr, "\pi"] \arrow[dr, "\psi_i"'] & & \mathcal{B}(\mathcal{H}) \\ & B_i \arrow[ur, "\chi_i"'] & \end{tikzcd}$$

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Theorem (MT)

(A, G, α) a C^*-dynamical system, G discrete. TFAE:

i. *there are (positive) Schur A-multipliers $(\varphi_i)_i$ such that:*
   a. $\sup_i \|\varphi_i\|_{\mathfrak{S}} \leq \infty$;
   b. *φ_i supported on $\{(s,t) : st^{-1} \in K_i$ a finite set$\}$;*
   c. *$\|\varphi_i(s,t)(\alpha_{s^{-1}}(a)) - \alpha_{s^{-1}}(a)\| \to 0$ uniformly on a strip ($a \in A$);*
   d. *the space $\{\varphi_i(s, r^{-1}s)(\alpha_{s^{-1}}(a)) : a \in A,\ s,t \in G\}$ is finite-dimensional;*

ii. *$A \rtimes_{\alpha,r} G$ is exact.*

The $(S_{\varphi_i})_i$ give external approximations of $A \rtimes_{\alpha,r} G$: they do not preserve the **diagonal pattern**.

Exactness and nuclearity

We can prove Ozawa's result on **exactness of discrete groups**.

Theorem (Ozawa '00)

G discrete group. The following are equivalent:

i. *$C_r^*(G)$ is exact;*

ii. *the uniform Roe algebra $\ell^\infty(G) \rtimes_{\beta,r} G$ is nuclear.*

Exactness and nuclearity

We can prove Ozawa's result on **exactness of discrete groups**.

Theorem (Ozawa '00)

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i. *$C_r^*(G)$ is exact;*
ii. *the uniform Roe algebra $\ell^\infty(G) \rtimes_{\beta,r} G$ is nuclear.*

Proof (MT).

(ii) $\implies$ (i) is trivial because $C_r^*(G) \subset \ell^\infty(G) \rtimes_{\beta,r} G$.
(i) $\implies$ (ii) Take positive Schur multipliers $(\varphi_i : G \times G \to \mathbb{C})_i$ giving exactness. Identify with

$$\phi_i : G \to \ell^\infty(G);\ \phi_i(r)(s) := \varphi_i(s, r^{-1}s).$$

These are Herz–Schur $(\ell^\infty(G), G, \beta)$-multipliers which satisfy our nuclearity conditions. $\square$

Thank you for listening!