# Groupoid Banach algebras

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# Groupoids

$X$ **locally compact Hausdorff** topological space.

$\mathcal{G}$ topological groupoid on $X \subseteq \mathcal{G}$:

- set $\mathcal{G}$ of arrows $X \to X$;
- maps **domain** $d : \mathcal{G} \to X$ and **range** $r : \mathcal{G} \to X$;
- for $\gamma, \eta \in \mathcal{G}$ **composition** $\gamma\eta$ when $d(\gamma) = r(\eta)$;
- inverse $\gamma^{-1}$ for each $\gamma \in \mathcal{G}$;
- $X \subseteq \mathcal{G}$ as **units**;
- topology on $\mathcal{G}$ making operations continuous.

**Assumption**: $\mathcal{G}$ is étale: $d$ and $r$ **local homeomorphisms**.

Topology on $\mathcal{G}$ is **not** necessarily Hausdorff.

**Example**: Discrete group action $h : G \to \mathrm{Homeo}(X)$: $\mathcal{G} = X \times G$,

$$X \cong \{(x, e) : x \in X\}, \quad d(x, s) := x, \quad r(x, s) := h_s(x),$$

product $(h_s(x), t)(x, s) := (x, ts)$, inverse $(x, s)^{-1} := (h_s(x), s^{-1})$.

**Example**: Inverse semigroup action $h : S \to \mathrm{PHomeo}(X)$.

- $S$ inverse semigroup: semigroup $S$ (set with assoc. binary op);
- every $s \in S$ has unique generalised inverse $s^* \in S$: $ss^*s = s$, $s^*ss^* = s^*$;
- a **partial homeomorphism** on $X$ is a homeomorphism between open subsets of $X$: $h : d(h) \to r(h)$;
- partial homeomorphisms on $X$ is inverse semigroup $\mathrm{PHomeo}(X)$: composition on suitable domain, $h^* := h^{-1}$;
- **action** of $S$ on $X$ is semigroup homomorphism $h : S \to \mathrm{PHomeo}(X)$; $h_t : X_{t^*} \to X_t$;
- copy groupoid structure from previous example (+ **quotient**).

Note: Similarly inverse semigroup action on other objects by partial isomorphisms.

$\mathcal{G}$ a groupoid on $X$.

A bisection of $\mathcal{G}$ is **open set** $U \subseteq \mathcal{G}$ such that $r|_U$, $d|_U$ injective.

**Example**: $\mathcal{G} = X \rtimes_h G$ then we have bisections $U_t = X \times \{t\}$, $t \in G$.

Set of bisections $\mathrm{Bis}(\mathcal{G})$ is an **inverse semigroup**:

$$UV := \{\gamma\eta : \gamma \in U,\ \eta \in V\}, \quad U^* := \{\gamma^{-1} : \gamma \in U\}.$$

Get an inverse semigroup action $h : \mathrm{Bis}(\mathcal{G}) \to \mathrm{PHomeo}(X)$:

$$h_U : d(U) \to r(U);\ h_U := r \circ d|_U^{-1}.$$

**Example** (continued): $\mathcal{G} = X \rtimes_h G$ then $h_{U_t} = h_t$.

Inverse semigroup action $h : S \to \mathrm{PHomeo}(X) \rightsquigarrow$ groupoid $\mathcal{G}_h$.
Groupoid $\mathcal{G} \rightsquigarrow$ inv. semigroup action $h_{\mathcal{G}} : \mathrm{Bis}(\mathcal{G}) \to \mathrm{PHomeo}(X)$.
In particular $\mathcal{G} \cong X \rtimes \mathrm{Bis}(\mathcal{G})$.

# Groupoid algebras

- Well-developed theory of groupoid $C^*$-algebras;
- some other Banach algebras e.g. groupoid $L^p$-operator algebras and crossed product Banach algebras.

**Goal**: General construction of Banach algebra associated to $\mathcal{G}$; has the above as special cases.

Cover: twisted étale groupoid, not necessarily Hausdorff, $\mathbb{R}$- or $\mathbb{C}$-algebras, normed by family of representations.

Simplifications: no twist, only $\mathbb{C}$-algebras, Hausdorff groupoids, no choice of bisections.

$\mathcal{G}$ an étale groupoid with unit space $X$.

$$C_c(\mathcal{G}) := \mathrm{span}\big\{f \in C_c(U) : U \in \mathrm{Bis}(\mathcal{G})\big\}$$

is a $*$-algebra with operations

$$(f * g)(\gamma) := \sum_{\eta_1\eta_2=\gamma} f(\eta_1)g(\eta_2), \qquad f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

**Existing norms** on $C_c(\mathcal{G})$:

$$\|f\|_{L^1} := \max_{x\in X} \sum_{d(\gamma)=x} \big|f(\gamma)\big|, \qquad \|f\|_{L^\infty} := \max_{x\in X} \sum_{r(\gamma)=x} \big|f(\gamma)\big|,$$
$$\|f\|_I := \max\big\{\|f\|_{L^1}, \|f\|_{L^\infty}\big\}.$$

(Traditionally: a representation of $C_c(\mathcal{G})$ is a $\|\cdot\|_I$-contractive homomorphism...)

Recall that $f \in C_c(\mathcal{G})$, so $f = \sum_{U \in \mathrm{Bis}(\mathcal{G})} f_U$, where $f_U \in C_c(U)$.

**Observation**: the above norms all agree with supremum norm $\|\cdot\|_\infty$ on each $f_U$.

## Lemma (BKM)

*There is a maximal submultiplicative involutive norm $\|\cdot\|_{\max}$ on $C_c(\mathcal{G})$ which agrees with supremum norm $\|\cdot\|_\infty$ on each $f_U$, $U \in \mathrm{Bis}(\mathcal{G})$: for $f \in C_c(\mathcal{G})$*

$$\|f\|_{\max} := \inf \left\{ \sum_{k=1}^{n} \|f_{U_k}\|_\infty : f = \sum_{k=1}^{n} f_{U_k},\ f_k \in C_c(U_k),\ U_k \in \mathrm{Bis}(\mathcal{G}) \right\}.$$

$$\|f\|_\infty \leq \|f\|_{L^1}, \|f\|_{L^\infty} \leq \|f\|_I \leq \|f\|_{\max}$$

## Definition (BKM)

*The groupoid Banach algebra of $\mathcal{G}$ is $F(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{\max}}$.*

This is a Banach $*$-algebra.

# Representations

A representation of $F(\mathcal{G})$:

- in a Banach algebra $B$ is a contractive homomorphism $F(\mathcal{G}) \to B$;
- on a Banach space $E$ is a contractive homomorphism $F(\mathcal{G}) \to \mathcal{B}(E)$.

## Definition

*If $\mathfrak{r}$ is a family of representations of $F(\mathcal{G})$ then define*

$$F^{\mathfrak{r}}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{\mathfrak{r}}}, \quad \textit{where} \quad \|f\|_{\mathfrak{r}} := \sup_{\psi\in\mathfrak{r}} \|\psi(f)\|.$$

Examples:

- $\mathfrak{r} = \{\text{representations in } \mathcal{B}(\mathcal{H})\} \rightsquigarrow F^{\mathfrak{r}}(\mathcal{G}) = C^*_{\max}(\mathcal{G})$;
- 
  - $\mathfrak{r} = \{\text{representations in } \mathcal{B}(L^p(\mu))\} \rightsquigarrow F^{\mathfrak{r}}(\mathcal{G}) = F^p(\mathcal{G})$;
  - $\mathfrak{r} = \{\Lambda_p : C_c(\mathcal{G}) \to \mathcal{B}(L^p(\mathcal{G}))\} \rightsquigarrow F^{\mathfrak{r}}(\mathcal{G}) = F^p_r(\mathcal{G})$.

# Now what?

**Question**: Now we constructed $F^{\mathfrak{r}}(\mathcal{G})$, why is it useful?

I will try to convince you by showing several ways the construction reflects underlying data:

- underlying inverse semigroup action and representation theory;
- groupoid $L^p$-operator algebras;
- simplicity criteria for $F^{\mathfrak{r}}(\mathcal{G})$ reflected in $\mathcal{G}$.

# Representations and inverse semigroup actions

Recall: $\mathcal{G} \cong X \rtimes_h S$ for **inverse semigroup action** $S = \mathrm{Bis}(\mathcal{G})$:

$$h : S \to \mathrm{PHomeo}(X);\ \ h_t : X_{t^*} \to X_t,$$

where $X_s \subseteq X$ open.

Hence: inverse semigroup action

$$\hat{h} : S \to \mathrm{PAut}(C_0(X));\ \ \hat{h}_t : C_0(X_{t^*}) \to C_0(X_t),$$

given by $\big(\hat{h}_t(\phi)\big)(x) := \phi\big(h_t^{-1}(x)\big)$ where $C_0(X_s) \trianglelefteq C_0(X)$.

Sieben and Buss–Exel: crossed product $C^*$-algebras built from this data: completions of
$L^1(\hat{h}) := \{f \in L^1(S, C_0(X)) : f(t) \in C_0(X_{t^*})\}$ with product

$$(f * g)(r) := \sum_{st=r} \hat{h}_s\Big(\hat{h}_{s^*}\big(f(s)\big)g(t)\Big).$$

### Definition

$\hat{h} : S \to \mathrm{PAut}(C_0(X))$. *A covariant representation of* $\hat{h}$ *in* $B$ *is* $(\pi, v)$ *where*

$$\pi : C_0(X) \to B \text{ a representation} \quad \text{and} \quad v : S \to (B'')_1 \text{ a map,}$$

i. $v_t\pi(a) = \pi\big(\hat{h}_t(a)\big)v_t, \quad (a \in C_0(X_{t^*}))$;

ii. $\pi(a)v_s v_t = \pi(a)v_{st}, \quad (a \in C_0(X_{st}))$;

iii. $\pi(a)v_e = \pi(a), \quad (e \in E(S),\ a \in C_0(X_e))$;

iv. **normalisation**: $v_t = \lim_i \pi(e_i)v_t$, $((e_i)_i$ *approx. unit* $C_0(X_t))$.

Technical notes:

- in general $v_t \in B_1$ is too restrictive, we need the bidual;
- normalisation condition is equivalent to requiring each $v_t$ is a partial isometry, we can always add it;
- need to sort out relationship between $v_{t^*}$ and $(v_t)^*$;
- other normalisation: $B$ is a dual Banach algebra;
- representations on Banach space $E$ vs representations in $\mathcal{B}(E)$.

Covariant representations integrate to representations of $L^1(\hat{h})$ in usual way:

$$(\pi, v) \text{ covariant, } f \in L^1(\hat{h}) \rightsquigarrow \ \pi \rtimes v(f) := \sum_{t \in S} \pi\big(f(t)\big) v_t.$$

**Definition (BKM)**

*$\hat{h} : S \to \mathrm{PAut}(C_0(X))$ inverse semigroup action. $\mathfrak{r}$ a collection of covariant representations of $\hat{h}$. Define $\mathfrak{r}$-crossed product as*

$$C_0(X) \rtimes^{\mathfrak{r}}_{\hat{h}} S := \overline{L^1(\hat{h})}^{\|\cdot\|_{\mathfrak{r}}}, \quad \textit{where} \quad \|f\|_{\mathfrak{r}} := \sup_{(\pi,v) \in \mathfrak{r}} \|\pi \rtimes v(f)\|.$$

In particular $\mathfrak{r}$ **all** covariant representations gives universal crossed product $C_0(X) \rtimes_{\hat{h}} S$.

### Theorem (BKM)

*$\mathcal{G} \cong X \rtimes_h S$. There is a bijective correspondence between:*

- *representations $\psi : F(\mathcal{G}) \to B$;*
- **normalised** *covariant representations $(\pi, v)$ of $\hat{h}$ in $B$.*

*The correspondence is $\psi(a_t\delta_t) \leftrightsquigarrow \pi(a_t)v_t$ $(a_t \in C_c(X_t))$.*

Hence $F(\mathcal{G}) \cong C_0(X) \rtimes_{\hat{h}} S$. More generally

$$F^{\mathfrak{r}}(\mathcal{G}) \cong C_0(X) \rtimes_{\hat{h}}^{\mathfrak{r}} S$$

where $\mathfrak{r}$ all representations in $\mathcal{B}(E)$.

# Groupoid $L^p$-algebras

Previously studied by Gardella–Lupini, Choi–Gardella–Thiel, Hetland–Ortega, Phillips.

## Definition (BKM)

$p \in [1, +\infty]$. *Define the full groupoid $L^p$-operator algebra by*

$$F^p(\mathcal{G}) := F^{\mathfrak{r}}(\mathcal{G}),$$

*where $\mathfrak{r}$ is class of representations in $\mathcal{B}(L^p(\mu))$, $\mu$ localisable.*

Note: above is specific to $\mathbb{C}$-algebras; for $\mathbb{R}$-algebras need to modify.

This definition matches existing one of Gardella–Lupini.

We have seen that underlying representations of $F(\mathcal{G})$ are covariant representations of the inverse semigroup action $C_0(X) \rtimes_{\hat{h}} S$.

In $L^p$ case there are two types of data **underlying this**:

- spatial partial isometries (Phillips);
- $L^p$-partial isometries.

Underlying $(\Omega, \Sigma, \mu)$ is **Boolean algebra** $[\Sigma]$ — identify null sets. A set isomorphism is $\Phi : \Sigma \to \Sigma$ which descends to an isomorphism of Boolean algebras $[\Phi] : [\Sigma] \to [\Sigma]$. Subspace of $(\Omega, \Sigma, \mu)$ is $(D, \Sigma \cap D, \mu|_D)$, hence partial set automorphism. If $\Phi$ **partial set automorphism** then we get invertible partial map

$$T_\Phi : L^p(\mu) \to L^p(\mu); \;\; T_\Phi(1_C) = 1_{\Phi(C)}.$$

### Definition (Phillips)

*A spatial partial isometry is*

$$U_\Phi := \left( \frac{d\mu \circ \Phi^*}{d\mu} \right)^{1/p} T_\Phi, \quad U_\Phi \in \mathrm{PAut}(L^p(\mu)).$$

$E$ Banach space, $p \in [1, \infty]$. An $L^p$-projection on $E$ is $P \in \mathcal{B}(E)$ with:

$$P^2 = P, \quad \begin{cases} \|\xi\|^p = \|P\xi\|^p + \|(I-P)\xi\|^p & p < \infty; \\ \|\xi\| = \max\left\{\|P\xi\|, \|(I-P)\xi\|\right\} & p = \infty. \end{cases}$$

### Definition

*$E$ Banach space, $p \in [1, \infty]$. An $L^p$-partial isometry on $E$ is* **contraction** *$T \in \mathcal{B}(E)$ such that:*

- *there is contraction $T^* \in \mathcal{B}(E)$ with $TT^*T = T$ and $T^*TT^* = T^*$;*
- *$T^*T$ and $TT^*$ are $L^p$-projections.*

### Theorem (BKM)

*$p \in [1, \infty] \setminus \{2\}$. Spatial partial isometries on $L^p(\mu)$ coincide with $L^p$-partial isometries on $L^p(\mu)$.*

Now we understand the **underlying data** in $L^p$ case.

## Theorem (BKM)

*$p \in (1, \infty)$, $\mathcal{G} = X \rtimes_h S$. Every representation of $F^p(\mathcal{G})$ on $L^p(\mu)$ is a covariant representation $(\pi, v)$ where:*

- *$\pi$ represents $C_0(X)$ as* **multiplication operators** *and $v$ represents $S$ as* **spatial partial isometries***, equivalently as* $L^p$-**partial isometries***, on $L^p(\mu)$;*
- *equivalently, $\pi : C_0(X) \to L^\infty(\mu)$ and there is an inverse semigroup $\{\Phi_t\}_{t \in S}$ of partial set automorphisms such that $v_t = U_{\Phi_t}$.*

**Notes**:

- Important tool is Banach–Lamperti Theorem: for $p \in [1, \infty] \setminus \{2\}$ every (partial) isometry on $L^p(\mu)$ is **spatial**.
- Phillips **assumed** representations are spatial, we now know this is automatic.

# Simplicity

**Question**: When is $F^{\mathfrak{r}}(\mathcal{G})$ a simple Banach algebra?

Aim to imitate $C^*$-algebraic results. For example:

Theorem (Brown–Clark–Farthing–Sims)

*For* **Hausdorff** *second-countable $\mathcal{G}$ TFAE:*

(i) *$\mathcal{G}$ is topologically principal and minimal;*

(ii) *$C_r^*(\mathcal{G})$ is simple.*

Note: we will replace topologically principal by topologically free which is weaker (Kwaśniewski–Meyer).

$\mathcal{G}$ is called:

- **topologically free** if there is no non-empty open $V \subseteq \mathcal{G} \setminus X$ with $r|_V = d|_V$.
- **minimal** if there is no non-trivial open $U \subseteq X$ with $d(\gamma) \in U$ implies $r(\gamma) \in U$, $\gamma \in \mathcal{G}$.

We need the reduced $C^*$-algebra $C_r^*(\mathcal{G})$ here: the kernel of $\Lambda : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ is an ideal.

$C_0(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_\infty}$ space of **bounded Borel functions**.

## Lemma (BKM)

*For* **reasonable** $F^{\mathfrak{r}}(\mathcal{G})$ *the inclusion* $C_c(\mathcal{G}) \hookrightarrow C_0(\mathcal{G})$ *extends to the* **Renault map** $j_{\mathfrak{r}} : F^{\mathfrak{r}}(\mathcal{G}) \to C_0(\mathcal{G})$*:*

- $j_{\mathfrak{r}}$ *is a* $(C_c(\mathcal{G}), *)$*-bimodule map;*
- $\ker j_{\mathfrak{r}}$ *is a closed ideal of* $F^{\mathfrak{r}}(\mathcal{G})$*.*

## Definition

*Reasonable* $F^{\mathfrak{r}}(\mathcal{G})$ *is reduced if* $j_{\mathfrak{r}}$ *is* **injective***.*
$F_r^{\mathfrak{r}}(\mathcal{G}) := F^{\mathfrak{r}}(\mathcal{G}) / \ker j_{\mathfrak{r}}$ *is a reduced groupoid Banach algebra.*

This covers existing cases:

- reduced $C^*$-algebra $C_r^*(\mathcal{G})$ (Renault);
- reduced $L^P$-algebras $F_r^p(\mathcal{G})$ (Austad–Ortega, Phillips).

## Theorem (BKM)

*For* **Hausdorff** $\mathcal{G}$ *TFAE:*

(i) $\mathcal{G}$ *is topologically free and minimal;*

(ii) *every reduced reasonable* $F_r^{\mathfrak{r}}(\mathcal{G})$ *is simple;*

(iii) $F_r^1(\mathcal{G})$ *or* $F_r^{\infty}(\mathcal{G})$ *is simple;*

(iv) *a reduced reasonable* $F_r^{\mathfrak{r}_0}(\mathcal{G})$ *is simple and contains* $C_0(X)$ *as a maximal abelian subalgebra.*

Special cases of this result:

- $C^*$-algebra result above;
- simplicity criteria for $L^p$-Cuntz algebras (Phillips);
- simplicity of Cuntz–Krieger and graph algebras (Cortiñas–Rodriguez);
- crossed products by (inverse semi)group actions.

If $\mathcal{G}$ is **non-Hausdorff** we need to modify above construction.

$\mathfrak{M}_0(\mathcal{G})$: functions in $C_0(\mathcal{G})$ with meagre support.

### Lemma (BKM)

*For* **reasonable** $F^{\mathfrak{r}}(\mathcal{G})$ *the map* $C_c(\mathcal{G}) \hookrightarrow C_0(\mathcal{G})/\mathfrak{M}_0(\mathcal{G})$ *extends to the* **essential Renault map** $j^e_{\mathfrak{r}} : F^{\mathfrak{r}}(\mathcal{G}) \to C_0(\mathcal{G})/\mathfrak{M}_0(\mathcal{G})$*:*

- $j^e_{\mathfrak{r}}$ *is a* $(C_c(\mathcal{G}), *)$*-bimodule map;*
- $\ker j^e_{\mathfrak{r}}$ *is a closed ideal of* $F^{\mathfrak{r}}(\mathcal{G})$*.*

### Definition

*Reasonable* $F^{\mathfrak{r}}(\mathcal{G})$ *is essential if* $j^e_{\mathfrak{r}}$ *is* **injective***.*
$F^{\mathfrak{r}}_e(\mathcal{G}) := F^{\mathfrak{r}}(\mathcal{G})/\ker j^e_{\mathfrak{r}}$ *is an essential groupoid Banach algebra.*

The simplicity theorem (almost — delete (iv)) works if we replace **reduced** by **essential**.

Thank you for listening!